MATHEMATICS

Grade 11

Part - III

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The National Anthem of Sri Lanka

Sri Lanka Matha
Apa Sri Lanka Namo Namo Namo Namo Matha
Sundara siri barinee, surendi athi sobamana Lanka
Dhanya dhanaya neka mal palaturu piri jaya bhoomiya ramya
Apa hata sepa siri setha sadana jeewanaye matha
Piliganu mena apa bhakthi pooja Namo Namo Matha
Apa Sri Lanka Namo Namo Namo Namo Matha
Oba we apa vidya
Obamaya apa sathya
Oba we apa shakthi
Apa hada thula bhakthi
Oba apa aloke
Apage anuprane
Oba apa jeevana we
Apa mukthiya oba we
Nava jeevana demine, nithina apa pubudukaran matha
Gnana veerya vadawamina regena yanu mana jaya bhoomi kara
Eka mavakage daru kela bevina
Yamu yamu vee nopama
Prema vada sema bheda durerada
Namo, Namo Matha
Apa Sri Lanka Namo Namo Namo Namo Matha

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ආනන්ද සමරකෝන්

ஒரு தாய் மக்கள் நாமாவோம்
ஒன்றே நாம் வாழும் இல்லம்
நன்றே உடலில் ஓடும்
ஒன்றே நம் குருதி நிறம்

அதனால் சகோதரர் நாமாவோம்
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நலமே வாழ்தல் வேண்டுமன்றோ

யாவரும் அன்பு கருணையுடன்
ஒற்றுமை சிறக்க வாழ்ந்திடுதல்
பொன்னும் மணியும் முத்துமல்ல - அதுவே
யான்று மழியாச் செல்வமன்றோ.

ஆனந்த சமரக்கோன்
கவிதையின் பெயர்ப்பு.

Foreword

With the continuous advancement of the world, the education sector too is transformed. Therefore, if we require the creation of a student community who could confront the future challenges successfully, our learning teaching process must constantly utilize effective approaches. It is our responsibility to disseminate the knowledge of the new world while assisting to create global citizens with good values. Our department is actively engaged in producing learning tools with the great aim of contributing to enlighten the minds of the children of the country.

A textbook is a repository of knowledge. At times, it takes us to a world of entertainment while developing our critical thinking faculties. It promotes our hidden potentials. In the coming years, the memories related to these textbooks will bring you happiness. While making the maximum use of this valuable learning tool, you must essentially access other useful knowledge spaces too. This textbook is offered to you free of charge as a great gift of the free education. Only you can add a value to the great fortune that has been spent by the government to print these textbooks. I wish that you would gain the ability to enlighten the future as citizens with knowledge and values by using this textbook.

I would like to bestow my sincere thanks on the panels of writers, editors and reviewers as well as on the staff of the Educational Publications Department for the contribution made on this endeavor.

P.N. Ilapperuma,
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26. 06. 2020

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Illustrations, Cover Page and Technical Assistance

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Message of the Board of Compilers

This textbook has been compiled in accordance with the new syllabus to be implemented from 2015.

Textbooks are compiled for students. Therefore, we have made an attempt to compile this textbook in a simple and detailed manner making it possible for you to read and understand it on your own.

We have included descriptions, activities and examples to introduce the subject concepts in an attractive manner and to establish them. Moreover, activities are organized from simple to complex to develop an interest to do them.

We have used the terms related to mathematical concepts in accordance with the glossary of technical terms of mathematics compiled by the Department of Official Languages.

Some subject matter learnt during the earlier grades is necessary to learn the subject content in the grade 11 syllabus. Thus, review exercises are included at the beginning of each chapter to revise previous knowledge. You will be prepared by them for the subject content of grade 11.

In addition, students may use the grade 10 book which you have if you need to recall previous knowledge.

You will gain maximum benefit from this textbook by reading the chapters and doing the review exercises of each chapter even before your teacher teaches them in the classroom.

We hope that studying mathematics will be an interesting, joyful and productive experience.

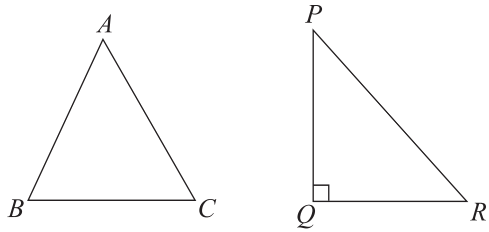
Board of Compilers

17 Pythagoras' Theorem

By studying this lesson you will be able to

- identify Pythagoras' Theorem
- use Pythagoras' Theorem in calculations and to prove riders
- identify Pythagorean triples.

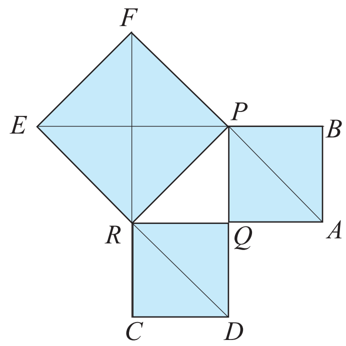
Introduction



The triangles ABC, PQR and KLM in the above figure are respectively an acute angled triangle, a right angled triangle and an obtuse angled triangle. These triangles have been thus named, by considering the largest (one or more) of the interior angles. Accordingly, the right angle $P\hat{Q}R$ is the largest interior angle of the triangle PQR. The side PR which is directly opposite this angle is the longest side of the triangle. This side is called the hypotenuse, and the remaining sides, namely PQ and QR, are known as the sides that include the right angle.

There is evidence to show that from ancient times, man knew about the geometrical properties of triangles. The marvel of the pyramids made in Egypt around 3000 B.C. is accepted by us all. For such creations, knowledge of geometry, especially on the characteristics of triangles is essential. In the "Rhind Papyrus" of around 1650 B.C. too, the main shape that can be observed is the triangle. Using this knowledge on the geometry of triangles, in 600 B.C., the Greek mathematician Pythagoras presented a special relationship between the lengths of the sides of right angled triangles. Although there is evidence to show that this relationship was known by early civilizations in countries such as China and India, Pythagoras is considered to be the first to offer a geometrical proof of this relationship. Later on in 300 B.C., the mathematician named Euclid included this result as a theorem, together with its proof, in his historical book called **THE ELEMENTS**.

17.1 Pythagoras' Theorem



A part of a floor on which tiles of the same shape and size have been placed is depicted in the above figure. The shape of each tile is an isosceles right angled triangle. Let us consider the isosceles right angled triangle *PQR*. The square *PQAB* has been drawn on the side *PQ* and the square *RCDQ* has been drawn on the side *RQ* (regions shaded in blue) of this triangle. The square drawn on the side *PQ* has an area equal to that covered by two tiles. Similarly, the square drawn on the side *QR* also has an area equal to that covered by two tiles, while the square *PREF* drawn on the hypotenuse *PR* has an area equal to that covered by four tiles.

Accordingly, for the squares lying on the three sides of the isosceles right angled triangle *PQR*, it is clear that the following relationship holds.

Area of square *PQAB* + Area of square *RCDQ* = Area of square *PREF*

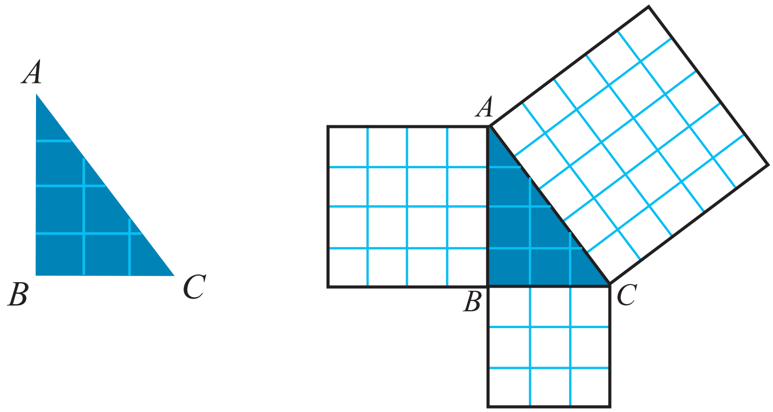
This relationship can be established further by doing the activity given below.

Activity

Using a square ruled paper, cut out 3 square shaped laminas and 1 triangular shaped lamina as follows.

(i) A square shaped lamina with side length equal to the length of 3 small squares
(ii) A square shaped lamina with side length equal to the length of 4 small squares
(iii) A square shaped lamina with side length equal to the length of 5 small squares
(iv) A right triangular shaped lamina, where the sides which include the right angle are of lengths equal to the length of 3 small squares and 4 small squares respectively.

Paste the right triangular shaped lamina on a white paper. Paste the squares on the three sides of the triangle as shown in the figure given below.



The area of the square on the side AB of the right angled triangle ABC	}	= 16 small squares
The area of the square on the side BC		= 9 small squares
The area of the square on the side AC		= 25 small squares
Accordingly, the sum of the areas of the squares on the sides which include the right angle of the triangle ABC	}	= 16 + 9 small squares
		= 25 small squares
The area of the square on the hypotenuse AC of the right angled triangle ABC	}	= 25 small squares

Therefore, in the right angled triangle ABC, the sum of the areas of the squares on the sides which include the right angle is equal to the area of the square on the hypotenuse of the triangle.

This relationship between the sides of a right angled triangle, which was known from ancient days, can be expressed as a theorem as follows.

Pythagoras' Theorem: In a right angled triangle, the area of the square drawn on the hypotenuse is equal to the sum of the areas of the squares drawn on the remaining sides of the triangle, which include the right angle.

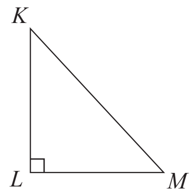
In the right angled triangle KLM shown in the figure, KM is the hypotenuse while KL and LM are the remaining sides which include the right angle.

Area of the square drawn on the side $KL = KL^2$

Area of the square drawn on the side $LM = LM^2$

Area of the square drawn on the hypotenuse $KM = KM^2$

Therefore, according to Pythagoras' Theorem,



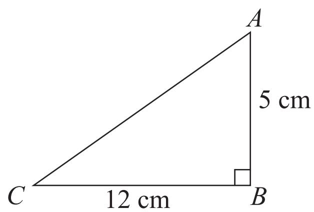
$$KL^2 + LM^2 = KM^2$$

Let us now consider how calculations are performed using Pythagoras' Theorem.

Example 1

In the right angled triangle ABC, $\hat{B} = 90^\circ$, $AB = 5$ cm and $BC = 12$ cm. Find the length of the side AC.

According to Pythagoras' Theorem,



$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 5^2 + 12^2 \\ &= 25 + 144 \\ &= 169 \end{aligned}$$

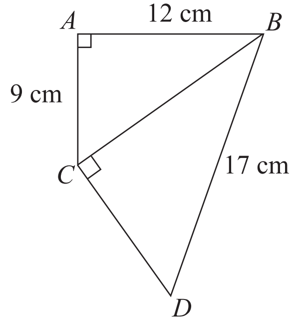
$$\therefore \quad AC = \sqrt{169} = 13$$

$\therefore$ the length of side AC is 13cm.

Example 2

Find the length of *CD* based on the information given in the figure.

According to the figure, Pythagoras' Theorem can be applied to the right angled triangle *ABC*.



$$\therefore\ BC^2 = AB^2 + AC^2$$
$$= 12^2 + 9^2$$
$$= 144 + 81$$
$$= 225$$
$$\therefore\ BC = \sqrt{225}$$
$$= 15$$

Applying Pythagoras' Theorem to the right angled triangle *BCD*,

$$CD^2 + BC^2 = BD^2$$
$$CD^2 + 15^2 = 17^2$$
$$CD^2 + 225 = 289$$
$$\therefore\ CD^2 = 289 - 225$$
$$= 64$$
$$\therefore\ CD = 8$$

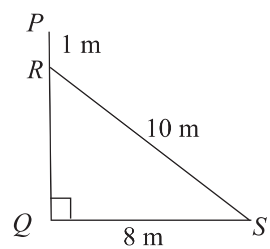
$\therefore$ the length of *CD* is 8 cm.

Now let us consider how Pythagoras' Theorem can be used to solve practical problems.

Example 3

One end of a wire is tied to a ring fastened at a point 1m below the top of a vertical utility pole, while the other end is tied to a ring fastened on the ground, 8 m away from the foot of the pole. The length of the wire between the two rings is 10 m. Find the height of the utility pole. (Assume that the wire is stretched.)

Let us draw the figure according to the given information.



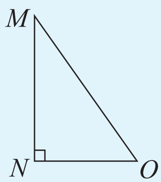
As the pole *PQ* is vertical, it makes a right angle with the horizontal ground. Therefore $P\hat{Q}S = 90^\circ$.
Since *QRS* is a right angled triangle, according to Pythagoras' Theorem,

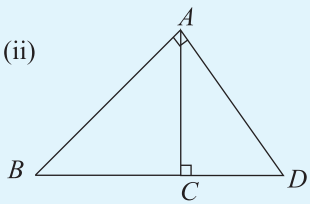
$$QR^2 + QS^2 = RS^2$$
$$QR^2 + 8^2 = 10^2$$
$$QR^2 + 64 = 100$$
$$\therefore \quad QR^2 = 100 - 64$$
$$QR^2 = 36$$
$$\therefore \quad QR = 6$$
$$\therefore \quad \text{Height of the pole} = QR + PR$$
$$= 6 + 1$$
$$= 7$$
$\therefore$ Height of the pole is 7 m.

Now do the following exercise using Pythagoras' Theorem.

Exercise 17.1

1. Fill in the blanks using the information in the figure.

(i)


(ii)


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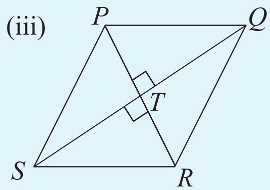

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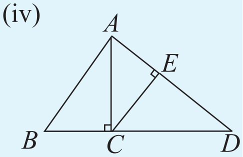


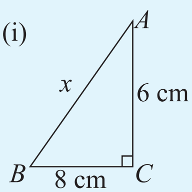
(i) $MO^2 = ... + ...$

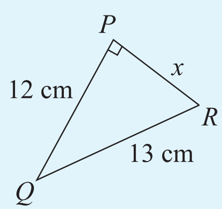
(ii) $BD^2 = ... + ...$
$.... = AC^2 + CD^2$
$AB^2 = AC^2 + ...$

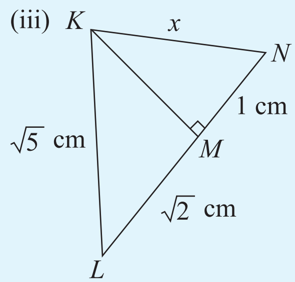
(iii) $PQ^2 = ... + ...$
$QR^2 = ... + ...$

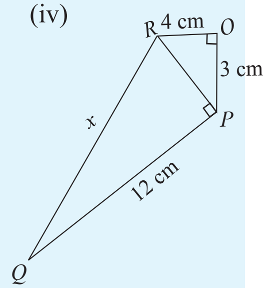
(iv) $AB^2 = ... + AC^2$
$.... = AE^2 + EC^2$
$AD^2 = AC^2 + ...$

2. Find the value of x in each of the right angled triangles given below.

(i)


(ii)


(iii)


(iv)


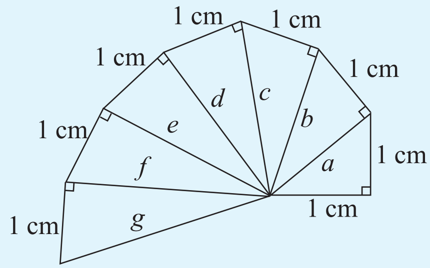
3. In the equilateral triangle ABC, D is the foot of the perpendicular drawn from the vertex A to the side BC. If the length of a side of the triangle is 2 cm, find the length of AD (Express the answer as a surd.)

4. The location Q is reached from the location P on the horizontal ground, by travelling 15m to the North from P and then 8m to the East.

(i) Draw a sketch based on the above information.
(ii) Find the distance PQ.

5. The lengths of the diagonals of a rhombus are 12 m and 16 m. Find the length of each side of the rhombus.

6. The figure illustrates the special creation Archimedes' spiral. By considering the right angled triangles in the figure, find the lengths a, b, c, d, e, f and g, using the given measurements. (Express the answers in surd form)



17.2 Further applications of Pythagoras' Theorem

Now let us consider how riders related to Pythagoras' Theorem are proved.

Example 1

$ABCD$ is a square. Prove that $AC^2 = 2AB^2$.

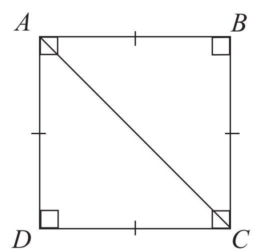
Proof : ABC is a right angled triangle since $A\hat{B}C = 90^\circ$

Applying Pythagoras' Theorem to the triangle ABC,

$AC^2 = AB^2 + BC^2$

$AC^2 = AB^2 + AB^2$ ($AB = BC$, sides of a square)

$\therefore \; AC^2 = 2AB^2$



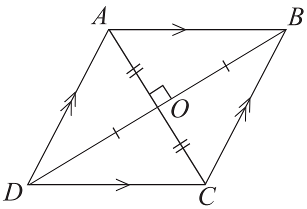
Example 2

In the rhombus $ABCD$, the diagonals AC and BD intersect at O. Prove that $AC^2 + BD^2 = 4\,AB^2$.

Proof: Since $ABCD$ is a rhombus, the diagonals bisect each other perpendicularly. (See figure)

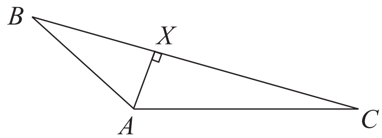
$\therefore \; A\hat{O}B = 90^\circ$, $AO = OC$ and $BO = OD$.

According to Pythagoras' Theorem; in the right angled triangle AOB,



$$AO^2 + OB^2 = AB^2$$
$$(\tfrac{1}{2}AC)^2 + (\tfrac{1}{2}BD)^2 = AB^2$$
$$\tfrac{1}{4}AC^2 + \tfrac{1}{4}BD^2 = AB^2$$
$$\tfrac{1}{4}(AC^2 + BD^2) = AB^2$$
$$\therefore \quad \underline{\underline{AC^2 + BD^2 = 4\,AB^2}}$$

Example 3



In the triangle ABC, $B\hat{A}C$ is an obtuse angle.
AX is drawn from A,
perpendicular to BC. Prove that $AB^2 - AC^2 = BX^2 - CX^2$

Proof:

In the right angled triangle AXB, according to Pythagoras' Theorem

$AB^2 = AX^2 + BX^2$ —— ①

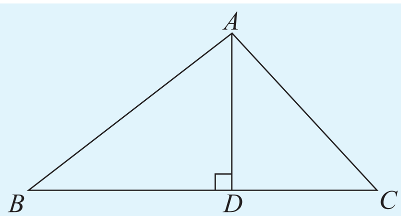
In the right angled triangle AXC, according to Pythagoras' Theorem,

$AC^2 = AX^2 + CX^2$ —— ②

① − ② ; $AB^2 - AC^2 = AX^2 + BX^2 - (AX^2 + CX^2)$
$= AX^2 + BX^2 - AX^2 - CX^2$
$= \underline{\underline{BX^2 - CX^2}}$

Prove the riders in the exercise given below as illustrated in the above examples.

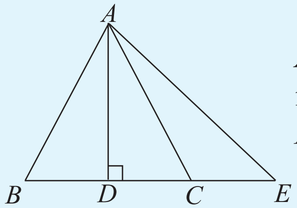
Exercise 17.2



1. AD is perpendicular to BC in the triangle ABC.
(See figure)
If $AD = DC$, prove that $AB^2 = BD^2 + DC^2$.

2. AD is perpendicular to BC in the triangle ABC. Prove that
$AB^2 + CD^2 = AC^2 + BD^2$.

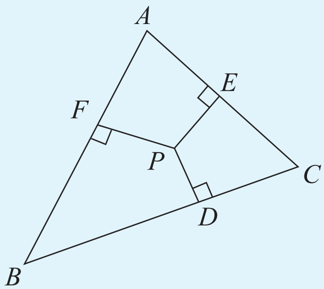
3. AD is perpendicular to BC in the equilateral triangle ABC. Prove that
$4\,AD^2 = 3\,BC^2$.

4. 

AD is perpendicular to BC in the equilateral triangle ABC in the figure. BC has been produced to E such that $DC = CE$. Prove that $AE^2 = 7\, EC^2$.

5. The diagonals of the quadrilateral $ABCD$ bisect each other perpendicularly at O. Prove that $AB^2 + CD^2 = AD^2 + BC^2$.

6. O is a point within the rectangle $ABCD$. Prove that $AO^2 + CO^2 = BO^2 + DO^2$ (Hint: Draw a parallel line through O to any side of $ABCD$.)

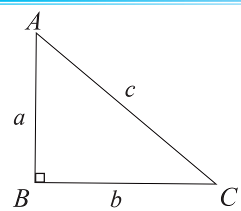
7. 

P is a point within the triangle ABC. The perpendiculars drawn from the point P to the sides BC, AC and AB meet these sides at D, E and F respectively.
Prove that,

(i) $BP^2 - PC^2 = BD^2 - DC^2$ and
(ii) $BD^2 + CE^2 + AF^2 = CD^2 + AE^2 + BF^2$

8. The two squares $ABXY$ and $BCPQ$ lie on the same side of the straight line ABC. Prove that $PX^2 + CY^2 = 3\,(AB^2 + BC^2)$

17.3 Pythagorean triples



In the right angled triangle ABC in the figure, if the lengths of the sides which include the right angle are a and b units, and the length of the hypotenuse is c units, then we know that $a^2 + b^2 = c^2$ according to Pythagoras' Theorem. Values of a, b and c which satisfy the equation

$a^2 + b^2 = c^2$ are known as Pythagorean triples.

Since $3^2 + 4^2 = 5^2$, we obtain that (3, 4, 5) is a Pythagorean triple. Any multiple of the triple (3, 4, 5) is also a Pythagorean triple.

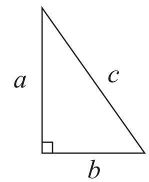
Eg: Multiplying each value of the triple (3, 4, 5) by 2 we obtain (6, 8, 10). Since $6^2 + 8^2 = 10^2$, (6, 8, 10) is a Pythagorean triple.

Multiplying each value of the triple (3, 4, 5) by 3 we obtain (9, 12, 15). Since $9^2 + 12^2 = 15^2$, (9, 12, 15) is also a Pythagorean triple.
There are Pythagorean triples apart from the multiples of (3, 4, 5).

Eg: Since $5^2 + 12^2 = 13^2$, (5, 12, 13) is a Pythagorean triple.
Since $8^2 + 15^2 = 17^2$, (8, 15, 17) is a Pythagorean triple.

A mathematician named Euclid introduced "parametric equations" to find Pythagorean triples. Given any two numbers x and y, if $a = x^2 - y^2$, $b = 2xy$ and $c = x^2 + y^2$, then $a^2 + b^2 = c^2$, and hence (a, b, c) is a Pythagorean triple.

Eg: $x = 6$, and $y = 5$, then $a = x^2 - y^2 = 6^2 - 5^2 = 11$
$b = 2xy = 2 \times 6 \times 5 = 60$
$c = x^2 + y^2 = 6^2 + 5^2 = 61$.
Therefore (11, 60, 61) is a Pythagorean triple.

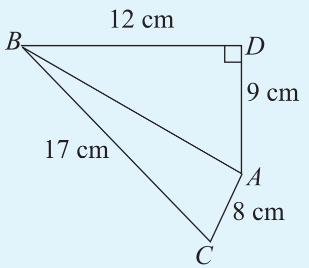


Exercise 17.3

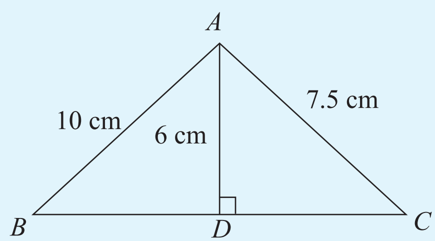
1. The following triples are the lengths of the sides of two triangles. Select the triangle which is a right angled triangle and write down the corresponding Pythagorean triple.

(i) (8, 15, 17) (ii) (14, 18, 25)

2. Based on the measurements given in figures (i) and (ii), show that $B\hat{A}C$ is a right angle in each figure.



(i)



(ii)

3. By completing the table given below, find the Pythagorean triples corresponding to the given pairs of values. Verify your answers.

x	y	x^2	y^2	a	b	c	Pythagorean triple
				$x^2 - y^2$	$2xy$	$x^2 + y^2$	
2	1						
5	4						
4	3						
6	5						
7	5						

Miscellaneous Exercise

1. The chord AB of the circle with centre O, which lies at a distance of 9 cm from O, is of length 24 cm. Find the radius of the circle.

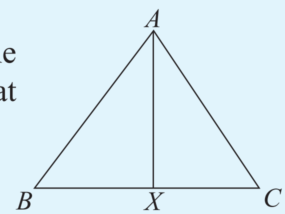
2. Construct the triangle ABC where $AB = 2$cm, $BC = 3$cm and $\hat{B}$ is a right angle. Using the triangle you constructed, find the value of $\sqrt{13}$ to the first decimal place.

3. Construct straight line segments of the lengths given below.

   (i) $\sqrt{8}$ cm (ii) $\sqrt{10}$ cm (iii) $\sqrt{41}$ cm

4. ABC is an equilateral triangle. D is the midpoint of AB and E is the midpoint of CD. Prove that $16\,AE^2 = 7\,AB^2$.

5. In the triangle ABC, $\hat{B}$ is an acute angle. The foot of the perpendicular dropped from A to BC is X. Prove that $AC^2 = AB^2 + BC^2 - 2\,BC.BX$



18 Trigonometry

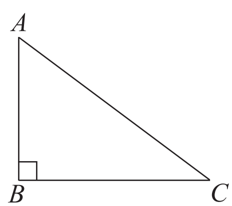
By studying this lesson you will be able to,

- identify the trigonometric ratios sine, cosine and tangent,
- perform calculations related to triangles using the sines, cosines and tangents tables,
- use the scientific calculator to examine the accuracy of the solutions to trigonometry problems.

18.1 Right Angled Triangles

We know that we can use Pythagoras' relationship to find the length of a side of a right angled triangle (right triangle) when the lengths of the other two sides are given.

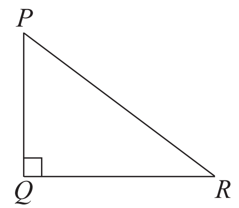
Pythagoras' relationship cannot be used to find the lengths of the remaining sides of a triangle when the length of one side of a right triangle and the magnitude of an angle other than the right angle are given. To identify a method to do this, let us first see how the sides of a right angled triangle are named.



The angle $\hat{B}$ of the right angled triangle ABC is a right angle. Therefore $\hat{A}$ and $\hat{C}$ are acute angles. The side AC which is opposite the right angle $\hat{B}$ is defined as the hypotenuse of the triangle. When the angle $\hat{C}$ is considered, then AB which is directly opposite $\hat{C}$ is called the "opposite side". Moreover, the side BC which is one of the arms of the angle $\hat{C}$, the other being the hypotenuse, is called the "adjacent side".

Accordingly, when $\hat{A}$ is considered, as above, we have that BC is the "opposite side" and AB is the "adjacent side".

Thus, for the triangle PQR in the figure,



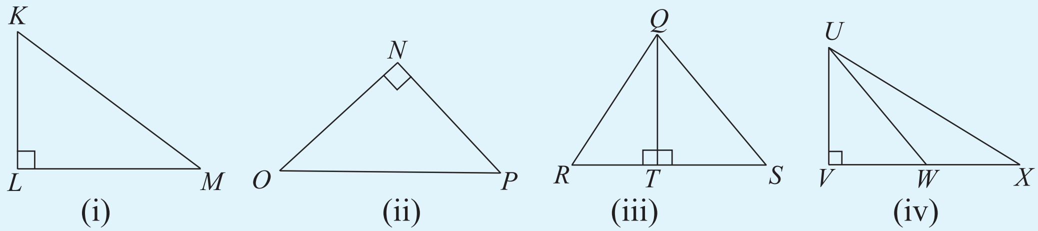
the hypotenuse $= PR$

when $Q\hat{R}P$ is considered, the opposite side $= PQ$
the adjacent side $= QR$

when $Q\hat{P}R$ is considered, the opposite side $= QR$
the adjacent side $= PQ$.

Exercise 18.1

1. Complete the following table using the given figures.



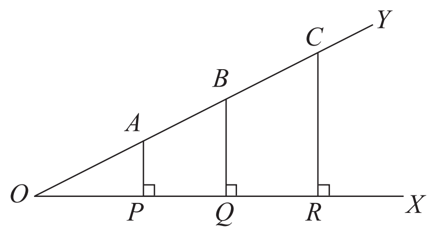
	Right angled triangle	Hypotenuse	Angle considered	Opposite side	Adjacent side
(i)	KLM	KM	$L\hat{K}M$ $L\hat{M}K$		
(ii)	PNO		$N\hat{O}P$ $O\hat{P}N$		
(iii)	QRT QTS		$R\hat{Q}T$ $T\hat{Q}S$		
(iv)	UVX UVW		$V\hat{U}X$ $U\hat{W}V$		

18.2 Trigonometric Ratios

Do the following exercise to investigate the relationship between two sides of a right angled triangle and an angle of the triangle.

Activity

- Draw the angle $X\hat{O}Y$ (= 30°) such that the arms XO and OY of this angle are of length 11 cm each.
- Mark the points A, B and C on the side OY at distances of 2 cm, 4 cm and 7 cm respectively from O.
- Using a set square or by some other method, draw lines perpendicular to OX through the points A, B and C and name the points that these perpendiculars meet OX as P, Q and R respectively.
- Then you will obtain a figure similar to the one given below.



- Measure the lengths of the sides of each of the right angled triangles that are obtained and complete the following table. (Write down all measured values and calculated values to the nearest first decimal place).

Right angled triangle	Hypotenuse	Side opposite 30°	Side adjacent to 30°	$\frac{\text{Opposite side}}{\text{Hypotenuse}}$	$\frac{\text{Adjacent side}}{\text{Hypotenuse}}$	$\frac{\text{Opposite side}}{\text{Adjacent side}}$
AOP	2	1	1.7	$\frac{1}{2} = 0.5$	$\frac{1.7}{2} = 0.9$	$\frac{1}{1.7} = 0.6$
BOQ						
COR						

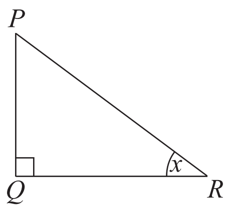
According to the above table which was prepared based on the measurements that were obtained from the activity,
for the angle 30°, from all three triangles we obtain

$$\frac{\text{opposite side}}{\text{hypotenuse}} = 0.5$$

$$\frac{\text{opposite side}}{\text{adjacent side}} = 0.6$$

$$\frac{\text{adjacent side}}{\text{hypotenuse}} = 0.9$$

Notice that the reason why a constant value is obtained for these ratios of sides of the right angled triangles in the above figure, is because the triangles are all equiangular. The above ratios which are defined for right angled triangles are called trigonometric ratios. These trigonometric ratios are called the sine of the angle 30°, the tangent of the angle 30° and the cosine of the angle 30°, depending on the sides connected with them. "sin" is used for sine, "tan" is used for tangent and "cos" is used for cosine. Accordingly, the sine of the angle 30° is written as sin 30°, the tangent of the angle 30° is written as tan 30° and the cosine of the angle 30° is written as cos 30°.



Now let us write the trigonometric ratios for the triangle *PQR* in the figure using the notation given above.

In terms of *x*;

$$\sin x = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{PQ}{PR}$$

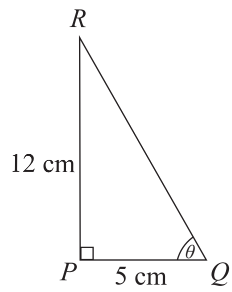
$$\cos x = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{QR}{PR}$$

$$\tan x = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{PQ}{QR}$$

Now let us see how calculations are done using these three trigonometric ratios by considering some examples.

Example 1

In the triangle *PQR* in the figure, $\hat{P}$ is a right angle. $PQ = 5$ cm and $PR = 12$ cm. $P\hat{Q}R = \theta$.



(i) Find the length of the side QR.
(ii) Find the following values.

(a) $\sin\theta$ **(b)** $\cos\theta$ **(c)** $\tan\theta$

(i) According to Pythagoras' relationship,

$$\begin{aligned} QR^2 &= PQ^2 + PR^2 \\ &= 5^2 + 12^2 \\ &= 25 + 144 \\ \therefore \quad QR &= \sqrt{169} \\ &= 13 \end{aligned}$$

$\therefore$ the length of the side QR is 13 cm.

(ii) **(a)** $\sin\theta = \dfrac{PR}{QR} = \dfrac{12}{13} = \underline{\underline{0.9230}}$

(b) $\cos\theta = \dfrac{PQ}{QR} = \dfrac{5}{13} = \underline{\underline{0.3846}}$

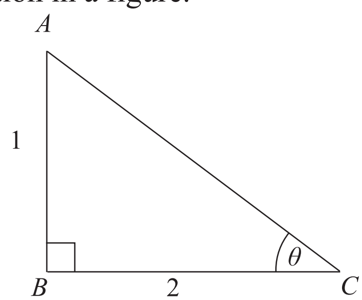
(c) $\tan\theta = \dfrac{PR}{PQ} = \dfrac{12}{5} = \underline{\underline{2.4}}$

Example 2

If $\tan\theta = \dfrac{1}{2}$, find the values of $\sin\theta$ and $\cos\theta$.

If $\tan\theta = \dfrac{1}{2}$, the side opposite θ is of length 1 unit and the side adjacent to θ is of length 2 units.

Let us represent this information in a figure.



Then, according to Pythagoras' relationship,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 1^2 + 2^2 \\ &= 5 \end{aligned}$$

$$\therefore \quad AC = \sqrt{5}$$

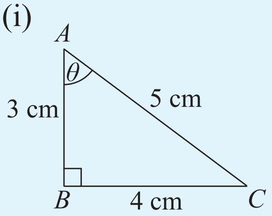
Then, $\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{1}{\sqrt{5}}$

$\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{2}{\sqrt{5}}$

Exercise 18.2

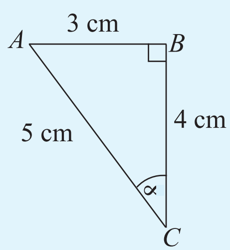
1. Fill in the blanks under each figure, based on the information given in each of the following figures.

(i)



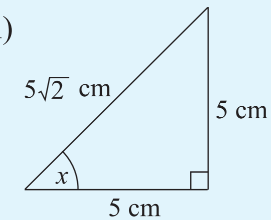
$\sin\theta = \ldots\ldots\ldots\ldots$
$\cos\theta = \ldots\ldots\ldots\ldots$
$\tan\theta = \ldots\ldots\ldots\ldots$

(ii)



$\sin\alpha = \ldots\ldots\ldots\ldots$
$\cos\alpha = \ldots\ldots\ldots\ldots$
$\tan\alpha = \ldots\ldots\ldots\ldots$

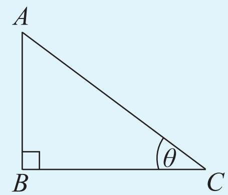
(iii)



$\sin x = \ldots\ldots\ldots\ldots$
$\cos x = \ldots\ldots\ldots\ldots$
$\tan x = \ldots\ldots\ldots\ldots$

2. If $\sin\theta = \dfrac{5}{13}$, find the value of (i) $\tan\theta$ (ii) $\cos\theta$.

3. In the triangle ABC in the figure, $\hat{B}$ is a right angle and $\hat{C} = \theta$.



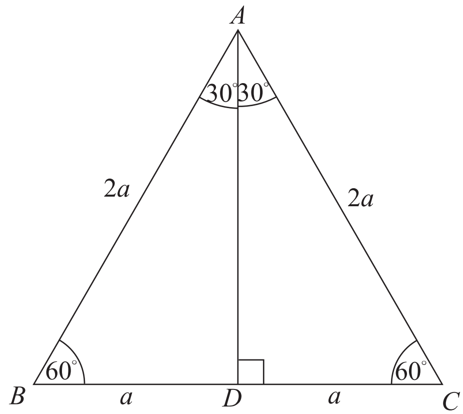
(i) Express $B\hat{A}C$ in terms of θ.

(ii) Show that $\sin\theta = \cos(90^\circ - \theta)$.

(iii) Show that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$.

18.3 The trigonometric ratios of the angles of magnitude 30°, 45° and 60°

By considering an equilateral triangle of side length $2a$, the trigonometric ratios for the angles 30° and 60° can be found.



The figure depicts an equilateral triangle ABC. Its vertex angles are each of magnitude 60°. We know that when the perpendicular AD from the vertex A to the side BC is drawn, D is the midpoint of BC and $B\hat{A}C$ is bisected by AD. Then $B\hat{A}D = 30^\circ$.

Let us find the length of the side AD of the right angled triangle ABD in terms of a.

According to Pythagoras' theorem;

$$\begin{aligned} BD^2 + AD^2 &= AB^2 \\ a^2 + AD^2 &= (2a)^2 \\ AD^2 &= 4a^2 - a^2 \\ &= 3a^2 \\ AD &= \sqrt{3}\,a \end{aligned}$$

If we consider the right triangle ABD,

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}\,a}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{\sqrt{3}\,a}{a} = \sqrt{3}$$

If we consider the right angled triangle *ABD,*

$$\sin 30^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

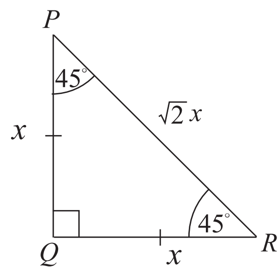
$$\cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{BD}{AD} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$

Let us now use the isosceles right angled triangle *PQR* in the given figure to obtain the trigonometric ratios for the angle 45°. Let us take the length of the sides of the triangle that include the right angle to be x.

Then, according to Pythagoras' theorem,

$$PR^2 = x^2 + x^2$$
$$= 2x^2$$
$$\therefore \quad PR = \sqrt{2}x$$



Accordingly,

$$\sin 45^\circ = \frac{PQ}{PR} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{QR}{PR} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{PQ}{QR} = \frac{x}{x} = 1$$

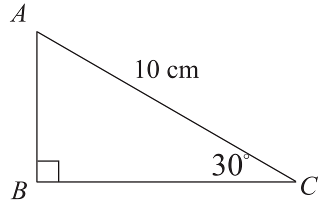
The trigonometric ratios obtained for the angles of magnitude 30°, 60° and 45° are given in the following table.

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Example 1

In the right angled triangle ABC, $\hat{B} = 90^\circ$, $A\hat{C}B = 30^\circ$ and $AC = 10$ cm. Find the lengths of AB and BC.

According to the figure,



$$\sin 30^\circ = \frac{AB}{AC}$$

$$\frac{1}{2} = \frac{AB}{10}$$

$$AB = 5$$

$\therefore$ the length of the side AB is 5 cm.

$$\cos 30^\circ = \frac{BC}{AC}$$

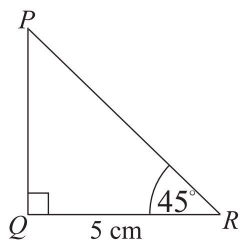
$$\frac{\sqrt{3}}{2} = \frac{BC}{10}$$

$$\therefore \quad BC = 5\sqrt{3}$$

$\therefore$ the length of the side BC is $5\sqrt{3}$ cm.

Example 2

Find the length of the hypotenuse of the right angled triangle PQR.



$$\cos 45^\circ = \frac{QR}{PR}$$

$$\frac{1}{\sqrt{2}} = \frac{5}{PR}$$

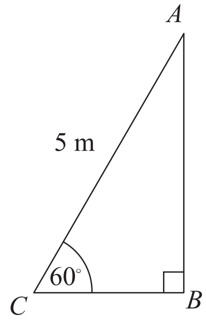
$$\therefore \quad PR = 5\sqrt{2}$$

$\therefore$ the length of the hypotenuse is $5\sqrt{2}$ cm.

Example 3

A 5 m long ladder is kept leaning against a vertical wall such that the angle between the ladder and the horizontal is 60°. At what height above the horizontal ground does the top of the ladder touch the wall?

Since the angle between the vertical wall and the horizontal ground is 90°, $A\hat{B}C = 90^\circ$ in the figure.



In the right angled triangle ABC,

$$\sin 60^\circ = \frac{AB}{AC}$$

$$\therefore \frac{\sqrt{3}}{2} = \frac{AB}{5}$$

$$\therefore AB = \frac{5\sqrt{3}}{2}$$

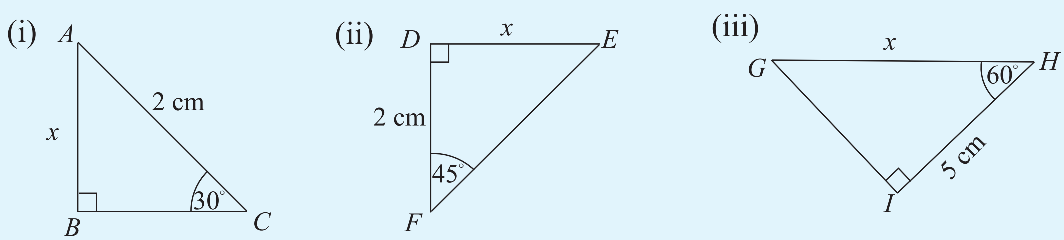
$$= 4.325 \text{ (By taking } \sqrt{3} = 1.73)$$

$\therefore$ the top of the ladder touches the wall at a height of 4.33 m above the ground.

Now do the following exercise by using the values in the table in section 18.3.

Exercise 18.3

1. Find the length x in each of the triangles given below using the information given in the triangle.



2. Find the value of each of the following expressions using the information in the table in section 18.3.

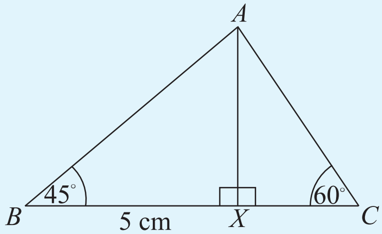
a. $\sin 30^\circ + \cos 60^\circ$
b. $\sin 45^\circ + \cos 45^\circ + \tan 60^\circ$
c. $\sin 60^\circ + \cos 30^\circ + \tan 60^\circ$
d. $\cos 60^\circ + \sin 30^\circ + \tan 60^\circ$

3. Verify the following.

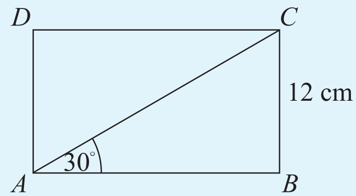
(i) $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ = 1$
(ii) $\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ = 0$

(iii) $\tan 30^\circ = \dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ}$

4. Based on the information in the figure,
(i) find the length of AX.
(ii) find the length of AC.
(Take $\sqrt{3} = 1.7$)



5. Find the length of the diagonal of the rectangle $ABCD$ in the figure if the length of the side BC is 12 cm.



6. To keep an antenna post vertical, one end of a stretched wire has been attached to a point which is 50 cm below the top of the post, while the other end has been attached to a wedge which is firmly fixed to the horizontal ground, 5 m away from the foot of the post. The angle between the horizontal ground and the wire is 30°.

(i) Represent this information in a sketch.
(ii) Find the height of the post by taking $\sqrt{3} = 1.7$

18.4 The Trigonometric Tables

So far we have considered only the trigonometric ratios of the angles 30°, 45° and 60°. However, there are trigonometric ratios for all angles from 0° to 90°. The trigonometric ratios corresponding to these angles have been tabulated. Separate tables have been prepared for sine, cosine and tangent. The "degree" which is a unit of angles can be divided into a smaller unit called "minute". One degree is equal to 60 minutes. i.e., $1^\circ = 60'$. Trigonometric ratio values for angles in degrees and minutes are provided in the trigonometric tables.

In all three tables, the sines, the cosines and the tangents, the first column has the angles from 0 to 90°. The following is a part of the Tangents Table.

ප්‍රකෘති ටැංජන
இயற்கைத் தான்சன்கள்
NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
0°	0·0000	0·0029	0·0058	0·0087	0·0116	0·0145	0·0175	89°	3	6	9	12	15	17	20	23	26
1	·0175	·0204	·0233	·0262	·0291	·0320	·0349	88	3	6	9	12	15	17	20	23	26
2	·0349	·0378	·0407	·0437	·0466	·0495	·0524	87	3	6	9	12	15	18	20	23	26
3	·0524	·0553	·0582	·0612	·0641	·0670	·0699	86	3	6	9	12	15	18	20	23	26
4	·0699	·0729	·0758	·0787	·0816	·0846	·0875	85	3	6	9	12	15	18	21	23	26

In the trigonometric ratios tables, the first column has the angles from 0° to 90° (Since only a portion of the table is given here, only the angles from 0° to 4° are shown). The parts of a degree which are minutes are given in the first row of the table as 0′, 10′, 20′ etc., and as 1′, 2′, ...9′ in the Mean Differences column. When finding the trigonometric ratio of an angle, the value in the relevant row and column, and sometimes the value in the Mean Differences column is used as is done when using the logarithms table.

Now let us consider each of the above mentioned trigonometric tables separately.

The Tangents Table

The ratios in the Tangents Table start with 0.0000, increase gradually, exceed 1.0000 and become extremely large as the angle approaches the magnitude 90°. Below is another portion of the Tangents Table.

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இயற்கைத் தான்சன்கள்
NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
42	.9004	.9057	.9110	.9163	.9217	.9271	.9325	47	5	11	16	21	27	32	37	43	48
43	.9325	.9380	.9435	.9490	.9545	.9601	.9657	46	6	11	17	22	28	33	39	44	50
44	.9657	.9731	.9770	.9827	.9884	.9942	1.0000	45	6	11	17	23	29	34	40	46	51
45	1.0000	1.0058	1.0117	1.0176	1.0235	1.0295	1.0355	44	6	12	18	24	30	36	41	47	53
46	.0355	.0416	.0477	.0538	.0599	.0661	.0724	43	6	12	18	25	31	37	43	49	55
47	.0724	.0786	.0850	.0913	.0977	.1041	.1106	42	6	13	19	26	32	38	45	51	57
48	.1106	.1171	.1237	.1303	.1369	.1436	.1504	41	7	13	20	27	33	40	46	53	60
49	.1504	.1571	.1640	.1708	.1778	.1847	.1918	40	7	14	21	28	34	41	48	55	62

Let us first find the value of tan 43°. The value corresponding to tan 43° appears in the row which contains 43° and the column which contains 0′. Accordingly, tan 43° = 0.9325.

Now let us find the value of tan 48° 20′

We need to move along the row containing 48° until we arrive at the column containing 20′. Take the value.1237 which is in this position. Since the whole number part of the number 1.0117 at the top of the column containing 20′ is 1, all the numbers along that column should also have a whole number part equal to 1. (The reason for writing the relevant whole number part only in the first row is to preserve the clarity of the tables). Accordingly, the value of tan 48° 20′ is 1.1237.

Let us find the value of tan 49° 57′ similarly. Here the value of tan 49° 50′ needs to be found first.

$$\tan 49^\circ\ 50' = 1.1847$$

To find the tangent value corresponding to 49° 57′, we need to add the value from the Mean Differences column, corresponding to 7′, which is 0.0048, to the value 1.1847 (as a convention, the mean difference is considered to be a value with four decimal places with only the non-zero part given in the tables).

Then we obtain

$$\tan 49^\circ\ 57' = 1.1847 + 0.0048$$
$$= 1.1895$$

Example 1

(i) $\tan 34^\circ\ 30' = 0.6873$

(ii) $\tan 44^\circ\ 42' = 0.9884 + 0.0011$
$= 0.9895$

(iii) $\tan 79^\circ\ 25' = 5.309 + 0.044$
$= 5.353$

When it is required to find an angle using the tables when a trigonometric ratio of the angle is known, a procedure similar to that followed in finding the antilog of a value using the logarithms table is used.

Let us find θ such that $\tan\theta = 1.1054$

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NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		Mean Differences (මධ්‍යන්‍ය අන්තරය / இடை வித்தியாசங்கள்) 1′	2′	3′	4′	5′	6′	7′	8′	9′
45°	1·0000	1·0058	1·0117	1·0176	1·0235	1·0295	1·0355	44	6	12	18	24	30	36	41	47	53
46	·0355	·0416	·0477	·0538	·0599	·0661	·0724	43	6	12	18	25	31	37	43	49	55
47	·0724	·0786	·0850	·0913	·0977	·1041	·1106	42	6	13	19	26	32	38	45	51	57
48	·1106	·1171	·1237	·1303	·1369	·1436	·1504	41	7	13	20	27	33	40	46	53	60
49	·1504	·1571	·1640	·1708	·1778	·1847	·1918	40°	7	14	21	28	34	41	48	55	62

Find the value in the table which is closest to 1.1054 but less than it. This value is 1.1041. From the table it can be seen that the angle corresponding to this value is $47^{\circ}\ 50'$. We need to add 0.0013 to 1.1041 to obtain the value 1.1054. Therefore, the number of minutes corresponding to 0.0013 (that is, the value 13 in the Mean Differences column) has to be added to $47^{\circ}\ 50'$ to obtain the correct angle. This is 2 minutes as highlighted in the table. Therefore, the angle of which the tangent value is 1.1054 is $47^{\circ}\ 50' + 2' = 47^{\circ}\ 52'$. Therefore, $\theta = 47^{\circ}\ 52'$.

Example 2

(i) If $\tan\theta = 0.3706$

$\theta = 20^{\circ}\ 20'$

(ii) If $\tan\theta = 0.4774$

$\theta = 25^{\circ}\ 30' + 1'$

$= 25^{\circ}\ 31'$

(iii) If $\tan\theta = 0.8446$

$\theta = 40^{\circ}\ 11'$

The Sines Table

This table contains values from 0.0000 to 1.0000. As in the Tangents Table, the first column contains the angles from 0° to 90°. In the first row right at the top of the table, the minute values, namely $0'$, $10'$, $20'$ etc., and in the Mean Differences column, the values $1'$, $2'$, ... $9'$ appear. This table is used in the same way that the Tangents Table is used.

Note: Although the values in the Tangents Table start from 0 and increase to very great values, the Sines Table contains only values from 0 to 1. The reason for this is because the sine value of an angle in a triangle always takes a value from 0 to 1.

Let us find the value of $\sin 33^{\circ}\ 27'$ using the table.

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இயற்கைச் சைன்கள்
NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
30°	0·5000	0·5025	0·5050	0·5075	0·5100	0·5125	0·5150	59	2	5	8	10	13	15	18	20	23
31	·5150	·5175	·5200	·5225	·5250	·5275	·5299	58	2	5	7	10	12	15	17	20	22
32	·5299	·5324	·5348	·5373	·5398	·5422	·5446	57	2	5	7	10	12	15	17	20	22
33	·5446	·5471	·5495	·5519	·5544	·5568	·5592	56	2	5	7	10	12	15	17	19	22
34	·5592	·5616	·5640	·5664	·5688	·5712	·5736	55	2	5	7	10	12	14	17	19	22

First note that $\sin 33^\circ 20' = 0.5495$. To obtain the value corresponding to the remaining $7'$, move along the row containing 33° and find the value corresponding to $7'$ from the Mean Differences column. This value is 0.0017. Add this value to 0.5495 to obtain the value of $\sin 33^\circ 27'$.

That is, $\sin 33^\circ 27' = 0.5495 + 0.0017 = 0.5512$.

Example 3

(i) $\sin 75^\circ\, 44' = 0.9689 + 0.0003$
$= 0.9692$

(ii) $\sin 45^\circ\, 34' = 0.7133 + 0.0008$
$= 0.7141$

(iii) $\sin 39^\circ\, 50' = 0.6406$

Now let us use the table to find the angle corresponding to a given sine value. This is also done in the same way that we found the angle corresponding to a given tangent value.

Let us find the angle θ if $\sin \theta = 0.5075$. It appear in the raw 30° and $30'$ column. accordingly if $\sin \theta = 0.5075$, the $\theta = 30^\circ\, 30'$.

Now let us find another angle using the table.
The angle θ if $\sin \theta = 0.5277$. Since 0.5277 does not appear in the table, consider the value 0.5275 which is the closest value in the table that is less than the given value. The angle corresponding to this is $31^\circ\, 50'$. Consider the values in the Mean Differences column along the same row to find the number of minutes corresponding to the remaining 0.0002. The number of minutes corresponding to the value 2 in the Mean Differences column is $1'$. $\therefore$ The angle of which the sine value is 0.5277 is $31^\circ\, 51'$.

That is, if $\sin \theta = 0.5277$, then $\theta = 31^\circ\, 51'$.

Example 4

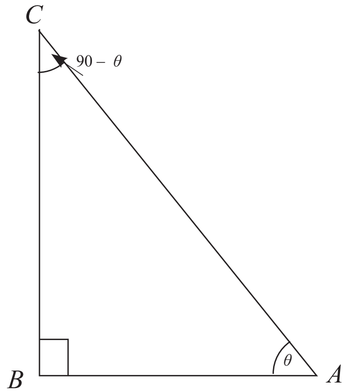
(i) If $\sin \theta = 0.5831$
then $\theta = 35^\circ\, 40'$

(ii) If $\sin \theta = 0.7036$
then $\theta = 44^\circ\, 43'$

(iii) If $\sin \theta = 0.9691$
then $\theta = 75^\circ\, 43'$

Cosines

Consider the following triangle.



The above triangle is a right angled triangle with $A\hat{B}C = 90^\circ$ Let us take $B\hat{A}C = \theta$. Then, since the sum of the angles of a triangle is 180°, $A\hat{C}B = 90^\circ - \theta$.

The sum of the angles $A\hat{C}B$ and $B\hat{A}C$ is 90°. You have learnt previously that such pairs of angles are called complementary angles.

If we consider the triangle ABC,

$$\cos\theta = \frac{\text{side adjacent to } \hat{A}}{\text{hypotenuse}} = \frac{AB}{AC}.$$

This can also be written as,

$$\sin(90^\circ - \theta) = \frac{\text{side opposite to } \hat{C}}{\text{hypotenuse}} = \frac{AB}{AC}.$$

Accordingly, $\cos\theta = \sin(90 - \theta)$.

This relationship can be used to find the cosine value of an angle in a triangle.

Example 1

Find the value of $\cos 58^\circ$.

$\cos 58^\circ = \sin(90^\circ - 58^\circ)$ (according to the relationship that was obtained)

$= \sin 32^\circ$

$= \underline{\underline{0.5299}}$ (according to the part of the table given above)

Example 2

Find the value of cos $56^\circ\ 18'$.

First let us find the value of $90 - 56^\circ\ 18'$. It is $33^\circ\ 42'$.
Therefore $\cos 56^\circ\ 18' = \sin (90 - 56^\circ\ 18') = \sin 33^\circ\ 42'$
$= \underline{\underline{0.5549}}$

We can similarly find the angle when the cosine value has been given. Let us consider an example.

Example 3

Find the value of θ if $\cos \theta = 0.5175$.
Let us write this as $\sin (90 - \theta) = 0.5175$.
Next let us find the angle of which the sine value is 0.5175. According to the table it is $31^\circ\ 10'$.

Therefore, $90 - \theta = 31^\circ\ 10'$.

The value of θ can be found by solving the above equation for θ.

Then $\theta = 90 - 31^\circ\ 10' = \underline{\underline{58^\circ\ 50'}}$

Note: The cosine value of an angle in a triangle, like the sine value of an angle in a triangle, always takes a value from 0 to 1. Apart from the above method, the cosine value of an angle in a triangle can also be found directly from the Sines Table. Observe that in the Sines Table, just before the Mean Differences column, there is a column with angle values which are obtained by subtracting the angle values in the first column from 90°. The cosine values of angles in a triangle can also be found by using the table values corresponding to the angle values in this column. However, in this case, the values in the Mean Differences column need to be subtracted instead of added.

Now let us consider how to use the table in relation to cosine values.

Let us find the value of cos $4^\circ\ 20'$ using the table.

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
80°	0.9848	0.9853	0.9858	0.9863	0.9868	0.9872	0.9877	9	0	1	1	2	2	3	3	4	4
81	.9877	.9881	.9886	.9890	.9894	.9899	.9903	8	0	1	1	2	2	3	3	3	4
82	.9903	.9907	.9911	.9914	.9918	.9922	.9925	7	0	1	1	2	2	2	3	3	3
83	.9925	.9929	.9932	.9936	.9939	.9942	.9945	6	0	1	1	1	2	2	2	3	3
84	.9945	.9948	.9951	.9954	.9957	.9959	.9962	5	0	1	1	1	1	2	2	2	3
85	0.9962	0.9964	0.9967	0.9969	0.9971	0.9974	0.9976	4									
86	.9976	.9978	.9980	.9981	.9983	.9985	.9986	3									
87	.9986	.9988	.9989	.9990	.9992	.9993	.9994	2									
88	.9994	.9995	.9996	.9997	.9997	.9998	.9998	1									
89	0.9998	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	0°									

(අන්තරය ඉතා කුඩා බැවින් වගු ගත කිරීම අනවශ්‍යය.)

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Example 4

We need to consider the row corresponding to 4° in the "degrees" column on the right hand side and the column corresponding to 20′ in the "minutes" column at the bottom. The value in the table in the row containing 4°, (found to the left), and in the column containing 20′ is 0.9971.

Therefore, cos 4° 20′ = 0.9971.

Example 5

Now let us find the value of cos 9° 26′.

We see that cos 9° 20′ = 0.9868 and that the value corresponding to 6′ is 0.0003.

Now to obtain the value of cos 9° 26′, the value obtained from the Mean Differences column has to be subtracted from 0.9868.

Accordingly,

$$\cos 9^\circ\ 26' = 0.9868 - 0.0003$$
$$= \underline{\underline{0.9865}}$$

Example 6

Now let us find the angle θ such that $\cos\theta = 0.4374$.

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
25	0.4226	0.4253	0.4279	0.4305	0.4331	0.4358	0.4384	64	3	5	8	10	13	16	18	21	24
26	.4348	.4410	.4436	.4462	.4488	.4514	.4540	63	3	5	8	10	13	16	18	21	23
27	.4540	.4566	.4592	.4617	.4643	.4669	.4695	62	3	5	8	10	13	15	18	21	23
28	.4695	.4720	.4746	.4772	.4797	.4823	.4848	61	3	5	8	10	13	15	18	20	23
29	.4848	.4874	.4899	.4924	.4950	.4975	.5000	60°	3	5	8	10	13	15	18	20	23

ප්‍රකෘති කෝසයින
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The value in the table closest to 0.4374 and less than it is 0.4358. The angle which has this value as its cosine is $64^{\circ}\ 10'$ according to the table. The value 0.0016 which is the difference between 0.4374 and 0.4358 is found in the column corresponding to 6'. This needs to be subtracted from $64^{\circ}\ 10'$.

$64^{\circ}\ 10' - 6' = 64^{\circ}\ 4'$

$\therefore$ the angle θ such that $\cos\theta = 0.4374$ is $64^{\circ}\ 4'$.

Exercise 18.4

1. Find each of the following values using the Tangents Table.

a. $\tan 25^{\circ}$ **b.** $\tan 37^{\circ}$ **c.** $\tan 40^{\circ}\ 54'$

2. Find the angle θ corresponding to each of the tangent values given below.

a. $\tan\theta = 0.3214$ **b.** $\tan\theta = 0.7513$ **c.** $\tan\theta = 0.9432$

3. Find each of the following values using the Sines Table.

a. $\sin 10^{\circ}\ 30'$ **b.** $\sin 21^{\circ}\ 32'$ **c.** $\sin 25^{\circ}\ 57'$

4. Find the angle θ corresponding to each of the sine values given below.

a. $\sin\theta = 0.5000$ **b.** $\sin\theta = 0.4348$ **c.** $\sin\theta = 0.6437$

5. Find each of the following values using the Cosines Table. Examine the accuracy of your answers by using the Sines Table.

a. $\cos 5^{\circ}\ 40'$ **b.** $\cos 29^{\circ}\ 30'$ **c.** $\cos 44^{\circ}\ 10'$

6. Find the angle θ corresponding to each of the cosine values given below.

a. $\cos\theta = 0.4358$ **b.** $\cos\theta = 0.6450$ **c.** $\cos\theta = 0.9974$

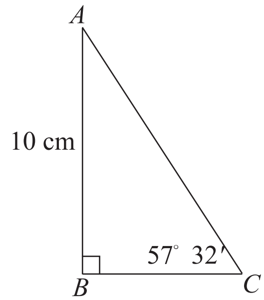
18.5 Solving problems using the trigonometric tables

The types of problems that were solved earlier involving the angles 30°, 45° and 60° only can now be solved for any angle in a triangle. It is important to take the following into consideration when solving problems related to trigonometry.

1. Consider a suitable right angled triangle.
2. Select a suitable angle in the triangle.
3. Use a suitable trigonometric ratio corresponding to the selected angle.

Now let us consider some examples.

Example 1



Find the length of the side AC by using the information in the triangle ABC in the figure.

The angle in the triangle that is given is C. The length that is given is of the side opposite this angle. The length of the hypotenuse needs to be found. $\therefore$ The trigonometric ratio which involves these two sides, namely the sine ratio needs to be used.

$$\sin 57^\circ\ 32' = \frac{AB}{AC}$$

$$0.8437 = \frac{10}{AC}$$

$$\therefore \quad AC = \frac{10}{0.8437}$$

Let us Find this value using the logarithms table.

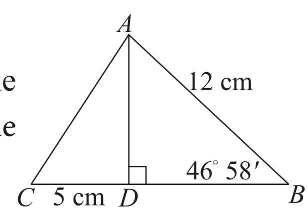
$$\text{Let } AC = \frac{10}{0.8437}.$$

$$\text{Then,} \quad \log AC = \log \frac{10}{0.8437}$$

$$= \log 10 - \log 0.8437$$
$$= 1 - \bar{1}.9262$$
$$= 1.0738$$
$$\therefore AC = \text{antilog } 1.0738$$
$$\therefore AC = 11.85$$

Therefore the length of AC (accurate to the second decimal place) is 11.85 cm.

Example 2



AD has been drawn perpendicular to the side BC of the triangle ABC. Find the magnitude of $A\hat{C}B$ using the information in the figure.

Here, the right angled triangle that needs to be considered to find the angle $A\hat{C}B$ is the triangle *ADC*. If the lengths of two sides of this triangle are known, then the required angle can be found. The length of one of its side, *CD* has been given as 5 cm. We need to find the length of one more side. Note that we can find the length of *AD* by considering the triangle *ABD*. Therefore, let us first find the length of *AD* by considering the triangle *ABD* and using the sine ratio.

$$\sin 46^\circ 58' = \frac{AD}{AB}$$

$$0.7310 = \frac{AD}{12}$$

$$12 \times 0.7310 = AD$$

$$\therefore \quad AD = 8.7720 \text{ cm}$$

Now, considering the right triangle *ACD*, $\tan A\hat{C}D = \dfrac{AD}{CD}$

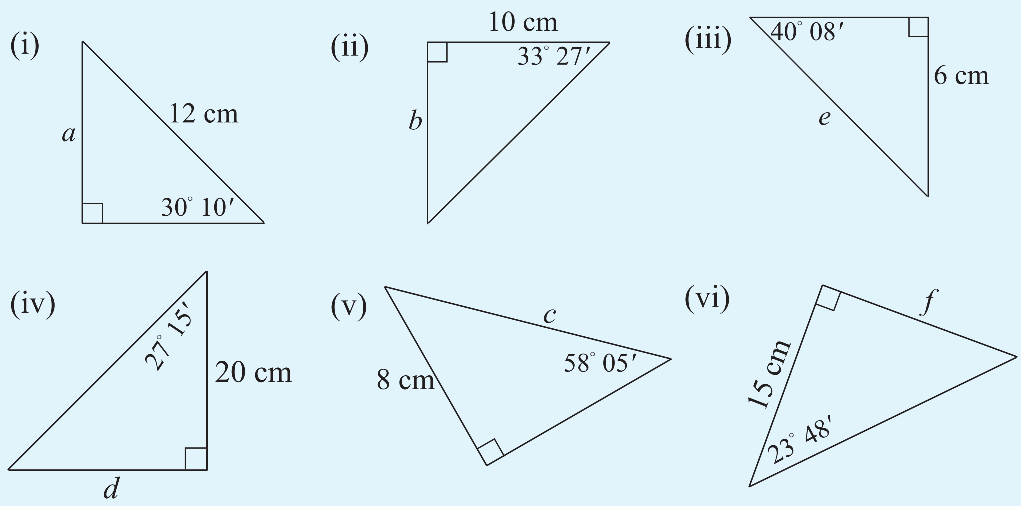
$$= \frac{8.7720}{5}$$

$$\therefore \tan A\hat{C}D = 1.7544$$

$$\therefore A\hat{C}D = 60^\circ 18'$$

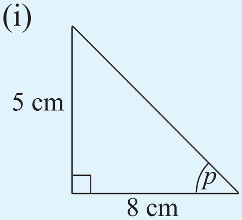
Exercise 18.5

1. Find the length denoted by an algebraic symbol in each of the following triangles.

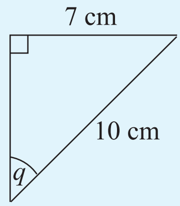


2. Find the angle denoted by an algebraic symbol in each of the following triangles.

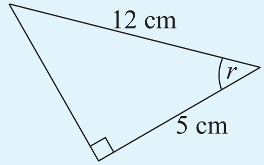
(i)



(ii)



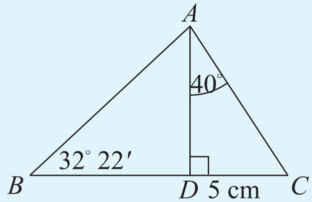
(iii)



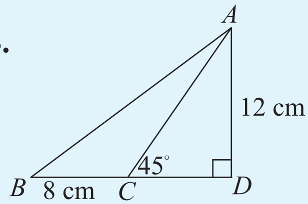
3. Based on the information in the given figure, find

(i) the perimeter of the triangle ABC.

(ii) the area of the triangle ABC.

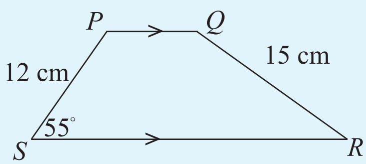


4.



Show using the information provided in the figure that the magnitude of the angle $A\hat{B}C$ of the triangle ABC, is $30^\circ\ 58'$.

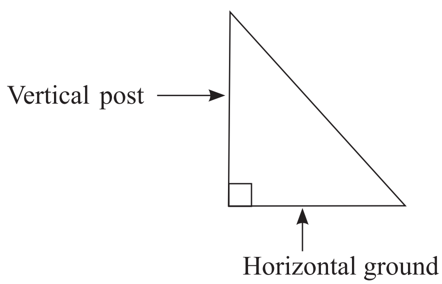
5.



In the trapezium $PQRS$, $SR > PQ$. If $PS = 12$ cm and $QR = 15$ cm, find the magnitude of $Q\hat{R}S$.

18.6 Angles in a vertical plane

A plane which is parallel to the earth (flat ground) is a horizontal plane. A plane which is perpendicular to a horizontal plane is a vertical plane. A post which is fixed perpendicular to the earth is a vertical post. Such a post is depicted in the figure.



You learnt in grade 10 to determine locations using scale diagrams involving angles of elevation and angles of depression. Now let us learn how to find using the trigonometric ratios.

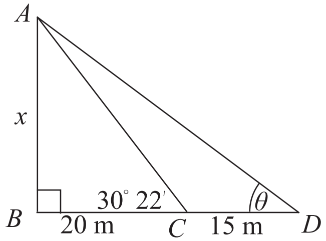
Let us consider the following example.

Example 1

A man is standing on the flat ground at a point C which is 20 m away from the foot of a vertical pillar AB. The angle of elevation of the top of the pillar from this point is $30^\circ\ 22'$. The man travels 15 m from this point along a straight path away from the pillar to another point, and again observes the top of the pillar.

(i) Represent this information in a rough sketch.
(ii) Find the height of the pillar to the nearest metre.
(iii) Find the angle of elevation of the top of the pillar from the second location.

(i)



(ii) Let us take the height of the pillar to be x metres.

Then, considering the right angled triangle ABC we obtain,

$$\tan 30^\circ\ 22' = \frac{AB}{BC}$$

$$\tan 30^\circ\ 22' = \frac{x}{20}$$

$$x = 20 \tan 30^\circ\ 22'$$
$$= 20 \times 0.5859$$
$$= 11.718$$

Therefore, the height of the pillar is approximately 12 m.

(iii) Let us take the angle of elevation of the top of the pillar from D to be θ.

Then, considering the right angled triangle ABD we obtain,

$$\tan \theta = \frac{AB}{BD}$$

$$\tan\theta = \frac{12}{35}$$

$$\tan\theta = 0.3428$$

$$\therefore\quad \theta = 18^{\circ}\,55'$$

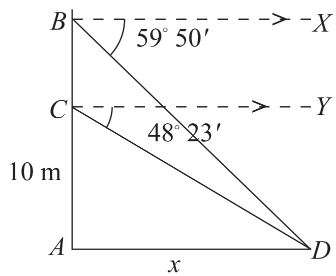
$\therefore$ the angle of elevation of the pillar from the second position is $18^{\circ}\,55'$.

Example 2

A person stands by a small window of a vertical building consisting of several floors. The window is located at a height of 10 m above the flat ground. The person observes a motorcycle which is parked on the flat ground a long distance from the building. The angle of depression of the motorcycle from the window is $48^{\circ}\,23'$. The person now ascends to the topmost storey of the building and observes the same motorcycle from another window. The angle of depression of the motorcycle when observed from this window is $59^{\circ}\,50'$.

(i) Represent this information in a sketch.
(ii) How far from the building is the motorcycle parked?
(iii) Calculate the height of the window in the topmost storey of the building from the flat ground, to the nearest second decimal place.

(i)



(ii) ACD in the figure is a right angled triangle. Let us take the distance from the building to the motorcycle to be x m.

Since $Y\hat{C}D = 48^{\circ}\,23'$, $A\hat{D}C = 48^{\circ}\,23'$ (alternate angles)

Therefore, by considering the right angled triangle ADC.

$$\tan 48^{\circ}\,23' = \frac{AC}{AD}$$

$$\tan 48^{\circ}\,23' = \frac{10}{x}$$

$$\therefore\quad \frac{10}{\tan 48^{\circ}\,23'} = x$$

Finding the value of x using the logarithms table

$\lg x = \lg 10 - \lg 1.1257$
$\quad = 1 - 0.0515$
$\therefore x = \text{antilog } 0.9485$
$\quad = 8.883$

That is, $x = \dfrac{10}{1.1257}$

$= 8.883$

$\therefore$ the distance from the building to the motorcycle is 8.883 m.

(iii) In the right angled triangle ABD, $A\hat{D}B = 59^\circ\ 50'$.

$$\tan 59^\circ\ 50' = \frac{AB}{AD}$$

$$\tan 59^\circ\ 50' = \frac{AB}{8.883}$$

$$AB = 8.883 \times 1.7205$$

$$= 15.28$$

$\therefore$ the window in the topmost storey of the building is located at a height of approximately 15.28 metres from the flat ground.

Do the following exercise according to the above examples.

Exercise 18.6

1. Draw a sketch based on the given information.

(i) A is located at the top of a vertical post AB. A person stands on the flat ground at a distance of 20 m from the foot of this post. The angle of elevation of the top of the pillar from the position of the eyes of the person is $55^\circ\ 20'$. The person is 1.5 m tall.

(ii) A technician attending to repairs, seated at the top of a vertical telecommunication pillar of height 35 m, observes a vehicle parked on the flat ground a long distance from the foot of the pillar. The angle of depression of the vehicle from the position of the technician is 50°.

(iii) A person on the second floor of a vertical building, observes a lighthouse at a distance of 75 m from the building. The angle of elevation of the top of the lighthouse from his position is $27^\circ\ 35'$ and the angle of depression of the foot of the lighthouse is $41^\circ\ 15'$.

(iv) The angle of elevation of the top of a utility post from the location where a child stands is 30°. The child travels a distance of 25 m along a straight path towards the utility post. The angle of elevation of the top of the utility post from this position is 50°. (Nelect the height of the child).

2. A security guard looking out from a window at the top of a lighthouse of height 20 m observes a ship travelling in the sea. The angle of depression of the ship from the top window is $30^\circ\ 15'$. Calculate the distance of the ship from the lighthouse.

3. The angle of elevation of the top of a vertical post from a point on the flat ground at a distance of 20 m from the foot of the post is $35^\circ\ 12'$. It is required to fix a taut wire from a point on the flat ground at a distance of 20 m from the foot of the post, to the top of the post to keep the post vertical. Find the length of the wire required for this. (Assume that half a meter of wire is used up to tie the wire)

4. The angle of elevation of the top of a vertical utility post fixed to the flat ground is 50° when observed from a point on the flat ground a certain distance from the foot of the post. If the height of the post is 12 m, find the distance from the foot of the post to the point of observation. (Neglect the height of the observer).

5. Two vertical posts A and B are fixed to the flat ground, a distance of 200 m from each other. The angle of elevation of the top of B, from the top of A is $4^\circ\ 10'$, and the angle of depression of the foot of B from this location is $8^\circ\ 15'$.

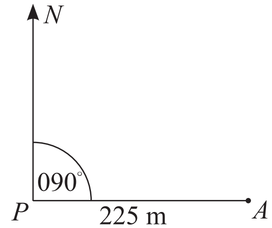
(i) Represent this information in a sketch.
(ii) Find the heights of the two posts A and B to the nearest metre.
(iii) Find the angle of elevation of the top of B from the foot of A.

6. A person stands right at the centre, between two vertical posts which are at a distance of 20 m from each other. The angle of elevation of the top of one post from this position is 60° while the angle of elevation of the top of the other post is 30° (neglect the height of the person).
(i) Find the heights of the two posts.
(ii) A taut wire has been drawn from the top of one post to the top of the other post. Find the length of the wire.

18.7 Angles in a horizontal plane

You have learnt earlier that bearings are used to indicate directions in a horizontal plane. Bearings provide a measure of an angle that is measured starting from the North and moving in a clockwise direction. Bearings are given using three digits. In modern measuring instruments the distance is also given with the bearing.

The point A which lies to the East of P, is located at a distance of 225 m from P on a bearing of 090°. This can be represented in a figure as follows.



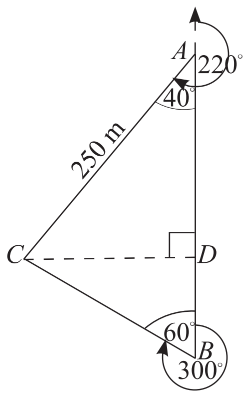
Let us see how calculations involving bearings are performed using trigonometric ratios by considering the following example.

Example 1

A straight road runs from the South to the North. When observed from a point A on this road, point C is located at a distance of 250 m on a bearing of 220°. When observed from another point B on the same straight road, C is located on a bearing of 300° .

(i) Represent the above information in a sketch.
(ii) Find the distance from C to the straight road.
(iii) Find the distance AB.

(i)



(ii) Since the bearing of C from A is 220°, $D\hat{A}C = 220° - 180°$

$= 40°$

Then by considering the right triangle ACD we obtain, $\sin 40° = \dfrac{CD}{AC}$.

$$AC \sin 40° = CD$$
$$CD = 250 \sin 40°$$
$$= 250 \times 0.6428$$
$$= 160.7000$$

$\therefore$ the shortest distance from C to the straight road AB is 160.7 m.

(iii) The length of $AB = AD + DB$

By considering the right triangle ACD, we obtain $\cos 40° = \dfrac{AD}{AC}$

$$AD = AC\cos 40^\circ$$
$$= 250 \times 0.7660$$
$$= 191.5000$$
$$= 191.5 \text{ m}$$

By considering the right triangle BDC, we obtain $\tan 60^\circ = \dfrac{CD}{DB}$

$$DB = \frac{CD}{\tan 60^\circ}$$
$$= \frac{160.7}{1.732}$$
$$= 92.78 \text{ m}$$

$\therefore$ the length of $AB = 191.5 + 92.78$ m
$= \underline{\underline{284.28 \text{ m}}}$

Exercise 18.7

1. Represent the following information in a sketch.

(i) B is located at a distance of 12 m from A on a bearing of 080° .

(ii) Q is located at a distance of 50 m from P on a bearing of 120°, and R is located at a distance of 25 m from Q on a bearing of 040°.

(iii) Y is located at a distance of 30 m from X on a bearing of 150°, Z is located at a distance of 100 m from Y on a bearing of 200° and A is located at a distance of 50 m from Z on a bearing of 080°.

2. A motorcyclist who starts a journey from a location A, travels 8 km to the East and then turns to the North and travels a further 6 km till he reaches the location B.

(i) Represent this information in a sketch.

(ii) Find the bearing of A from B.

(iii) Find the shortest distance between A and B.

3. A ship leaves harbour A and travels a distance of 150 km on a bearing of 040° until it reaches harbor B.

(i) How far towards the North is harbor B from harbour A?

(ii) How far towards the East is harbor B from harbour A?

4. A student who is trying to measure the breadth of a river which has straight parallel banks on the two sides, sits on one bank of the river and observes a tree, located directly in front of him on the opposite bank; the direction of the tree from the boy being perpendicular to the river banks. When the boy travels a distance of 75 m along the bank, he observes that the bearing of the tree from this location is 210°. Represent this information in a sketch and find the breadth of the river to the nearest metre using trigonometric ratios.

5. A group of forest conservationists observes from a distance that a fire has commenced in a forest. Using information they received on the location of the fire, starting from their camp *C*, they travel 2.5 km along a main road *A* on a bearing of 070° to the location P, and then from P by travelling 1.5 km on a bearing of 340°, they reach the location F of the fire.

(i) Represent this information in a figure.

(ii) Show with reasons that the group of conservationists was able to reach the location of the fire as quickly as possible, due to turning off the main road at *P*.

(iii) On what bearing would the conservationists have first observed the fire from their camp?

18.8 Using the calculator to find trigonometric ratios

When performing calculations involving trigonometric ratios using a scientific calculator, first the [MODE] key should be used to display "DEG" on the screen.

Let us see how these calculations are performed by considering some examples.

Example 1

Express using a flowchart how the keys of a calculator need to be activated to obtain the following values.

(i) tan 35° (ii) sin 35° (iii) cos 35°

(i) tan 35° [ON] — [tan] — [3] — [5] — [=] ⟶ [0.7002]

(ii) sin 35° [ON] — [sin] — [3] — [5] — [=] ⟶ [0.5736]

(iii) cos 35° [ON] — [cos] — [3] — [5] — [=] ⟶ [0.8192]

Example 2

Calculate the value of θ in each of the following cases.

(i) $\tan \theta = 1.2131$ (ii) $\sin \theta = 0.7509$ (iii) $\cos \theta = 0.5948$

(i) [ON] — [SHIFT] — [tan] — [1] — [.] — [2] — [1] — [3] — [1] — [=] ⟶ [50.5°]

(ii) [ON] — [SHIFT] — [sin] — [0] — [.] — [7] — [5] — [0] — [9] — [=] ⟶ [48.66°]

(iii) [ON] — [SHIFT] — [cos] — [0] — [.] — [5] — [9] — [4] — [8] — [=] ⟶ [53.5°]

Note: Observe that the value of each angle is obtained only in degrees.
For example $50.5^{\circ} = 50^{\circ}\,30'$

Exercise 18.8

1. Write down the order in which the keys of a calculator need to be activated to obtain the (i) tan value (ii) sin value (iii) cos value of the following angles.

a. 40° **b.** 75° **c.** 88° **d.** 43°

2. Express using a flowchart how the keys of a calculator need to be activated to obtain the value of θ in each of the following cases.

a. $\sin\theta = 0.9100$
b. $\sin\theta = 0.7112$
c. $\sin\theta = 0.1851$
d. $\cos\theta = 0.1853$
e. $\cos\theta = 0.7089$
f. $\cos\theta = 0.4550$
g. $\tan\theta = 0.5736$
h. $\tan\theta = 0.7716$
i. $\tan\theta = 0.9827$

Miscellaneous Exercise

1. Two ships P and Q leave a harbor simultaneously. Both ships travel at the same uniform speed of 18 kilometres per hour. P travels from the harbor on a bearing of 010° while Q travels on a bearing of 320°. Find the distance between the two ships after an hour.

2. Two tall buildings are located on opposite sides of a road. One building is 9 m taller than the other. The angle of elevation of the top of the shorter building from the foot of the taller building is $42^{\circ}\,20'$ If the shorter building is of height 15 m, determine the following. (Neglect the height of the observer)

(i) The distance between the two buildings.
(ii) The angle of elevation of the top of the taller building from the foot of the shorter building.

3. $AB = 10$ cm, $BC = 7$cm and $A\hat{B}C = 30^{\circ}\,26'$. The perpendicular drawn from A to BC is AX. Find the area of the triangle ABC.

4. Two flagpoles have been fixed on flat ground. Two points A and B are located along the straight line joining the feet of the two flagpoles. The angles of elevation of the two flagpoles from A are 30° and 60°, and from B are 60° and 45°. The distance between A and B is 10 m.

(i) Find the heights of the two flagpoles.
(ii) Find the distance between the two flagpoles.

19 Matrices

By studying this lesson you will be able to,

- identify a matrix
- identify the elements and the order of a matrix
- add and subtract matrices
- multiply a matrix by an integer
- multiply two matrices
- solve problems related to matrices.

19.1 Introducing Matrices

The idea of matrices was introduced in 1854 by the British mathematician Arthur Cayley. Let us identify matrices using a simple example.

The marks obtained by Wimal, Farook and Radha in a term test, for the subjects Mathematics and Science are shown in the table below.

	Mathematics	Science
Wimal	75	66
Farook	72	70
Radha	63	81

The numbers in the table given above can be represented in a matrix as follows.

$$\begin{pmatrix} 75 & 66 \\ 72 & 70 \\ 63 & 81 \end{pmatrix}$$

Here the columns indicate the subjects while the rows indicate the students. This information can also be represented in a matrix as follows.

$$\begin{pmatrix} 75 & 72 & 63 \\ 66 & 70 & 81 \end{pmatrix}$$

Here the columns indicate the students while the rows indicate the subjects.

An array of numbers organized in rows and columns is known as a matrix. Given below are several examples of matrices.

(i) $\begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

(iv) $\begin{pmatrix} -2 & 3 \\ 2 & 0 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 6 & 1 \\ 5 & 2 & 2 \end{pmatrix}$ (vi) $\begin{pmatrix} -5 & 4 \\ 9 & -1 \\ 0 & 4 \end{pmatrix}$

The numbers in a matrix are called the elements of the matrix. The elements of a matrix, apart from being numbers may also be algebraic symbols or expressions which stand for numbers.

Matrices are named using capital letters of the English alphabet. In instances when elements are expressed in terms of algebraic symbols, simple letters of the English alphabet are used.

Example 1

Three matrices which are named are shown below.

$A = \begin{pmatrix} 5 & 4 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$ $P = \begin{pmatrix} 2 & c \\ a & b \end{pmatrix}$

Example 2

In a Cartesian plane, the coordinates of the points A and B are (0, 5) and (4, 3) respectively. Represent this information in a matrix. Name it P.

In a table,

	A	B
x	0	4
y	5	3

As a matrix,

$$P = \begin{pmatrix} 0 & 4 \\ 5 & 3 \end{pmatrix}$$

Order of a Matrix

Consider the matrix $A = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 2 & 0 \end{pmatrix}$

The number of rows in A is 2 and the number of columns is 3. Using the number of rows and number of columns, we write the order of the matrix as 2×3. A is known as a "two by three" matrix.

Accordingly, A is sometimes written as,

$$A = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 2 & 0 \end{pmatrix}_{2 \times 3}.$$

Example 1

Write the order of each of the matrices given below.

(i) $\begin{pmatrix} 3 & 2 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}$

Number of rows in the matrix = 3
Number of columns in the matrix = 2
Order of the matrix = 3×2

(ii) $\begin{pmatrix} 3 & 2 & 4 \end{pmatrix}$

Number of rows = 1
Number of columns = 3
Order of the matrix = 1×3

(iii) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Number of rows = 2
Number of columns = 1
Order of the matrix = 2×1

(iv) $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Number of rows = 2
Number of columns = 2
Order of the matrix = 2×2

Row Matrices, Column Matrices and Square Matrices

Matrices with only one row are known as **row matrices**, matrices with only one column are known as **column matrices** and matrices which have an equal number of rows and columns are known as **square matrices**. Since the number of columns and the number of rows of a square matrix are equal, the order of a square matrix with two rows and two columns is known as a square matrix of order 2 and a matrix with three rows and three columns is known as a square matrix of order 3 etc.

For example,

$A = \begin{pmatrix} 3 & 2 & 5 \end{pmatrix}$ is a row matrix.

$B = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ is a column matrix.

$C = \begin{pmatrix} 5 & 2 & 1 \\ 3 & 1 & 3 \\ 0 & 4 & 0 \end{pmatrix}$ is a square matrix.

Identity Matrices and Symmetric Matrices

$$P = \begin{pmatrix} 3 & 2 & 4 \\ 6 & 5 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

In the square matrix given above, the main diagonal is highlighted. The string of elements from the top leftmost corner to the bottom rightmost corner is the main diagonal.

Note: The main diagonal is defined only for square matrices. Most often, the main diagonal is known simply as the diagonal.

The main diagonal of a square matrix of order two is highlighted below.

$$Q = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$$

Given below is a special square matrix.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The main diagonal of A consists of only the number 1. Apart from the diagonal elements, all the other elements are 0. This type of matrix is known as an **identity matrix**. The matrix A is the identity matrix of order 3 × 3. Given below is the identity matrix of order 2 × 2.

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

When naming identity matrices, the letter I is used. The identity matrix with n rows and n columns is written as $I_{n \times n}$. Accordingly,

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Can you identify the special feature in the matrix given below?

$$X = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

Consider the elements around the main diagonal of X. The elements around the main diagonal which are equal to each other are placed symmetrically about the main diagonal. Such matrices are called **symmetric matrices**.

$$Y = \begin{pmatrix} 1 & 5 \\ 5 & 3 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The elements around the main diagonals of Y and Z which are equal are placed symmetrically about the main diagonal. Therefore these are symmetric matrices too.

Note: Symmetric matrices are defined only for square matrices.

Exercise 19.1

1. Sarath bought 2 oranges and 3 mangoes, Kamal bought 4 oranges and 1 mango and Raju bought 1 orange and 5 mangoes from a certain fruit stall.

(i) Express the amount of fruits bought by Sarath as a row matrix.
(ii) Express the amount of fruits bought by Kamal as a row matrix.
(iii) Express the amount of fruits bought by Raju as a row matrix.
(iv) Construct a matrix with quantities of fruits Sarath, Kamal and Raju bought respectively as its rows.

2. Write the order of each of the following matrices.

(i) $A = \begin{pmatrix} 3 & 2 \\ 5 & 1 \\ 4 & 3 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & 2 \end{pmatrix}$ (iii) $C = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$

(iv) $D = \begin{pmatrix} 0 & 4 \end{pmatrix}$ (v) $E = \begin{pmatrix} 5 & 8 & 3 \end{pmatrix}$ (vi) $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

3. From the matrices given below, select the row matrices and the column matrices.

(i) $P = \begin{pmatrix} 3 & 0 & 2 \end{pmatrix}$ (ii) $Q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (iii) $R = \begin{pmatrix} 4 & 3 \end{pmatrix}$

(iv) $S = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ (v) $T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ (vi) $U = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}$

4. From the following matrices, select and write down the

(i) square matrices
(ii) symmetric matrices
(iii) identity matrices

Highlight the diagonals of the square matrices.

$A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 2 & 1 \\ 4 & 0 & 4 \\ 2 & 2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

19.2 Addition and Subtraction of Matrices

We have learnt how to add, subtract and multiply numbers. We have seen how easily we can solve problems by using these mathematical operations. We can similarly define mathematical operations on matrices too. Let us first consider how to add matrices.

Consider the two matrices A and B given below.

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 0 \\ 9 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 6 \\ 3 & 9 \\ 2 & 8 \end{pmatrix}$$

Both these matrices have the same order, 3 × 2. The addition of A and B is defined as the matrix which is obtained when the corresponding elements of A and B are added together.

Accordingly,

$$A + B = \begin{pmatrix} 4 & 1 \\ 2 & 0 \\ 9 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 6 \\ 3 & 9 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 5 & 9 \\ 11 & 13 \end{pmatrix}$$

By corresponding elements we mean the elements which are in the same position of the matrix. For example, the element in the first row and second column of the matrix A is 1. The corresponding element in matrix B is 6; that is the number in the first row and second column of B.

Now let us consider an example with algebraic symbols.

If $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ then, $X + Y = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 \\ x_3 + y_3 & x_4 + y_4 \end{pmatrix}$

The addition of matrices is defined only for matrices of the same order. Accordingly addition of matrices of different orders is not defined.
Now let us consider how the addition of two matrices is used through an example. Even though this example is a very simple one, it shows clearly how matrices can be used in practical situations.

Example 1

Praveen and Tharindu are two bowlers in the school cricket team. The number of wickets they took in the years 2014 and 2015 in one day and two day matches are shown in the tables given below.

	2014	2015
Praveen	21	23
Tharindu	15	16

Wickets taken in one day matches

	2014	2015
Praveen	14	16
Tharindu	9	19

Wickets taken in two day matches

Let us name the matrix which has the information on the one day matches as A and the one with the information on the two day matches as B.

Then we can write $A = \begin{pmatrix} 21 & 23 \\ 15 & 16 \end{pmatrix}$ and $B = \begin{pmatrix} 14 & 16 \\ 9 & 19 \end{pmatrix}$

In these matrices the columns represent years while the rows represent wickets. Let us find $A + B$.

$$A + B = \begin{pmatrix} 35 & 39 \\ 24 & 35 \end{pmatrix}$$

Think about what the matrix $A + B$ represents. It shows the total number of wickets Praveen and Tharindu took in 2014 and 2015 in both one day and two day matches. It can be represented in a table as shown below.

	2014	2015
Praveen	35	39
Tharindu	24	35

Total wickets taken

The subtraction of one matrix from another is also defined similarly. Here, the corresponding numbers are subtracted. For subtraction too, the orders of the two matrices should be equal. As an example,

$$A = \begin{pmatrix} 5 & 9 \\ 2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 4 \\ 6 & 0 \end{pmatrix} \text{ then, } A - B = \begin{pmatrix} 4 & 5 \\ -4 & 3 \end{pmatrix}.$$

Let us consider another example.

If X is the 3×3 matrix with all the elements equal to 2 and Y is the 3×3 identity matrix, then find the matrix $X - Y$.

$$X = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$X - Y = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

Equality of two matrices

Let us consider what it means when we say that two matrices are equal.

$$A = \begin{pmatrix} 2 & 3 \\ 10 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

For A and B to be equal matrices $a = 2$, $b = 3$, $c = 10$ and $d = 9$.
That is, for two matrices to be equal, their orders should be equal and the corresponding elements of the two matrices should be equal.

Exercise 19.2

1. Simplify the following matrices.

(i) $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$

(ii) $\begin{pmatrix} 3 & -2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & -2 & -4 \end{pmatrix}$

(iii) $\begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

(iv) $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

(v) $\begin{pmatrix} 2 & -2 \\ 3 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ -2 & 1 \\ 2 & -2 \end{pmatrix}$

(vi) $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}$

(vii) $\begin{pmatrix} 2 & 5 & -1 \\ 3 & 4 & 6 \\ 2 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -4 & 4 \\ -4 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix}$

(viii) $\begin{pmatrix} 5 & 4 & 2 \\ 2 & 3 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 5 \\ 5 & 3 & 10 \end{pmatrix}$

2. Simplify the following matrices.

(i) $\begin{pmatrix} 4 & 3 \\ 2 & 5 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} -3 \\ 5 \\ 8 \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix}$

(iii) $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} - \begin{pmatrix} -2 & 3 \\ 5 & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} 5 & -3 & -2 \end{pmatrix} - \begin{pmatrix} 2 & -4 & -2 \end{pmatrix}$

(v) $\begin{pmatrix} 5 & 3 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$ (vi) $\begin{pmatrix} 6 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 2 & -2 \\ 2 & 0 & 2 \\ 1 & -5 & -4 \end{pmatrix}$

3. If $\begin{pmatrix} 2 & 3 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} a & b & c \end{pmatrix}$ find the values of a, b and c.

4. If $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find the values of a, b, c and d.

5. If $\begin{pmatrix} 5 & 3 & 2 \\ 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} x & 2 & -1 \\ y & 1 & z \end{pmatrix} = \begin{pmatrix} 8 & 5 & 1 \\ 2 & 2 & 3 \end{pmatrix}$ find the values of x, y and z.

6. If $\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} x & 3 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ find the values of x and y.

19.3 Multiplication of a matrix by a number

Now let us consider what it means to multiply a matrix by a number. Multiplying a matrix by a number means multiplying each element in the matrix by that number. The matrix obtained by multiplying matrix A by the number k is written as kA. Let us only consider multiplying by an integer. For example, let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & 8 & 1 \end{pmatrix}$$

When A is multiplied by 5 the matrix obtained is,

$$5A = \begin{pmatrix} 5 \times 3 & 5 \times 1 & 5 \times 0 \\ 5 \times (-2) & 5 \times 8 & 5 \times 1 \end{pmatrix} = \begin{pmatrix} 15 & 5 & 0 \\ -10 & 40 & 5 \end{pmatrix}.$$

When A is multiplied by -3 the matrix obtained is,

$$-3A = \begin{pmatrix} -3 \times 3 & -3 \times 1 & -3 \times 0 \\ -3 \times -2 & -3 \times 8 & -3 \times 1 \end{pmatrix} = \begin{pmatrix} -9 & -3 & 0 \\ 6 & -24 & -3 \end{pmatrix}.$$

Note: The matrix obtained by multiplying matrix A by a number k has the same order as A.

Example: If $X = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix}$, then find the matrix $3X - 2Y$.

$$3X - 2Y = 3\begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix} + (-2)\begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 12 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} -10 & 4 \\ -4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 16 \\ -1 & -2 \end{pmatrix}$$

Exercise 19.3

1. Simplify the following matrices.

(i) $3\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ (ii) $4\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (iii) $3\begin{pmatrix} 2 & -1 & 3 \\ -3 & 1 & 2 \end{pmatrix}$

(iv) $2\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ (v) $3\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ -3 & 2 & 0 \end{pmatrix}$ (vi) $-2\begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}$

2. Find a, b, c and d if $3\begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

3. Find the values of x, y and z if $4\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -12 \\ 2 \end{pmatrix}$

4. Find the values of x, y, a and b if $2\begin{pmatrix} 5 & x \\ -2 & 9 \end{pmatrix} - 3\begin{pmatrix} y & -5 \\ 4 & a \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ b & 0 \end{pmatrix}$

19.4 Multiplication of Matrices

From the above definitions you may have realized that addition and subtraction of matrices as well as multiplication of a matrix by a number are carried out as for numbers. However, multiplication of two matrices is done in a different way. This is shown below.

Initially let us look at how to multiply a row matrix and a column matrix. If A is a row matrix of order $1 \times m$ and B is a column matrix of order $m \times 1$, then the matrix AB is defined and is of order 1×1 . To describe how matrix multiplication is done in this case, let us consider an example.

Let us assume that $A = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Accordingly, the order of A is 1×2 while the order of B is 2×1. Then AB is defined as,

$$AB = (a_1b_1 + a_2b_2)_{\,1 \times 1}$$

Example 1

Find AB if $A = \begin{pmatrix} 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$AB = (5 \times 3 + 2 \times 1) = (17)$$

We learnt earlier that any matrix can be multiplied by a number. We also learnt that addition and subtraction of matrices however can be done only if the matrices are of the same order. Multiplication of matrices too can be done in some cases only. Above we saw how to multiply a row matrix and a column matrix. We can multiply matrices which are of different orders too. In general, AB is defined if the order of A is $m \times n$ and the order of B is $n \times p$; that is, if the number of columns of A is equal to the number of rows of B. In this case we get a matrix of order $m \times p$, which is the number of rows of A into the number of columns of B. Let us now consider how such products of matrices are found.

As an example, let us see how to find AB if $A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}_{2 \times 2}$ and $B = \begin{pmatrix} 1 & 8 \\ 6 & 7 \end{pmatrix}_{2 \times 2}$

When multiplying the above two matrices, multiply each row of A by each column of B, in the same way that we multiplied a row matrix by a column matrix.

$$= \begin{pmatrix} (2\ \ 4)\begin{pmatrix} 1 \\ 6 \end{pmatrix} & (2\ \ 4)\begin{pmatrix} 8 \\ 7 \end{pmatrix} \\ (3\ \ 5)\begin{pmatrix} 1 \\ 6 \end{pmatrix} & (3\ \ 5)\begin{pmatrix} 8 \\ 7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 2 \times 1 + 4 \times 6 & 2 \times 8 + 4 \times 7 \\ 3 \times 1 + 5 \times 6 & 3 \times 8 + 5 \times 7 \end{pmatrix}$$

$$= \begin{pmatrix} 26 & 44 \\ 33 & 59 \end{pmatrix} \text{ (by finding each product)}$$

The way the above matrix AB was obtained can be explained as below.

- The element in the first row and first column of AB is obtained by multiplying the first row of A (row matrix) by the first column of B (column matrix).
- The element in the first row and second column of AB is obtained by multiplying the first row of A (row matrix) by the second column of B (column matrix).
- The element in the second row and first column of AB is obtained by multiplying the second row of A (row matrix) by the first column of B (column matrix).
- The element in the second row and second column of AB is obtained by multiplying the second row of A (row matrix) by the second column of B (column matrix).

Any two matrices for which the product is defined (that is, the number of columns of A is equal to the number of rows of B), can be multiplied as above. Let us look at a few more examples.

Example 2

If $X = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ then show that XY is defined and find it. Is YX defined?

Number of columns in $X = 2$ and the number of rows in $Y = 2$. As the number of columns in X and the number of rows in Y are equal, XY is defined.

Now,

$$XY = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

By multiplying each row of X by each column of Y we obtain,

$$= \begin{pmatrix} (4 \;\; 6)\begin{pmatrix} 1 \\ 7 \end{pmatrix} \\ (2 \;\; 3)\begin{pmatrix} 1 \\ 7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 4 \times 1 + 6 \times 7 \\ 2 \times 1 + 3 \times 7 \end{pmatrix}$$

$$= \begin{pmatrix} 46 \\ 23 \end{pmatrix}$$

Now let us see whether YX is defined.
The number of columns in Y is 1 while the number of rows in X is 2. As the number of the columns in Y is not equal to the number of rows in X, YX is not defined.

Let $P = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 6 & 3 \end{pmatrix}$. Under this section on matrix multiplication we first defined the product of matrices of the form QP. This product can be found by using the above definition too. That is, by multiplying each row in Q by each column in P.

$$QP = \begin{pmatrix} 6 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = (9).$$

This is a matrix with just one element. A matrix with only one element can be considered as a number. Therefore we write $QP = 9$.

Furthermore PQ is also defined. The product of the matrices P and Q, namely PQ, is a 2×2 matrix.

$$PQ = \begin{pmatrix} 2 \\ -1 \end{pmatrix}\begin{pmatrix} 6 & 3 \end{pmatrix} = \begin{pmatrix} 2 \times 6 & 2 \times 3 \\ (-1) \times 6 & (-1) \times 3 \end{pmatrix} = \begin{pmatrix} 12 & 6 \\ -6 & -3 \end{pmatrix}$$

Exercise 19.4

1. Simplify the following matrices.

(i) $\begin{pmatrix} 3 & 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

(ii) $\begin{pmatrix} 3 & 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & -1 \\ 0 & -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 5 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$

(v) $\begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

(vi) $\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

(vii) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(viii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$

(ix) $\begin{pmatrix} 2 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

(x) $\begin{pmatrix} 2 & -3 \end{pmatrix} \times \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$

2. Find a and b if $\begin{pmatrix} 2 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix}$

3. A, B and C are three matrices. $A \times B = C$. Fill in the blanks in the following table.

Order of matrix A	Order of matrix B	Order of matrix C
1 × 2	2 × 1	....
2 × 2	.... × 1	....
....× 2	.... × 1	1 × 1
.... ×	1 ×	2 × 2
.... × 1	.... × 2	1 × .....

4. If $P = \begin{pmatrix} 2 & -1 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $R = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
find,

(i) $P \times Q$
(ii) $P \times R$
(iii) $Q \times R$.

5. If $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

(i) Find AB .
(ii) Find BA .
(iii) What is the relationship between AB and BA?

6. $C = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$

(i) Find CD .
(ii) Find DC .

20 Inequalities

By studying this lesson you will be able to

- solve inequalities of the form $ax + b \gtreqless cx + d$ and represent the solutions on a number line,
- express problems related to day to day activities as inequalities and solve them.

Do the review exercise given below to recall what was learnt in grade 10 about solving inequalities of the form $ax + b \gtreqless c$

Review Exercise

1. Solve each of the inequalities given below.

a. $3x - 2 > 4$	**b.** $\frac{x}{2} + 5 \leq 7$	**c.** $5 - 2x > 11$
d. $\frac{x}{-2} + 3 \leq 5$	**e.** $\frac{5x}{6} + 4 \geq 14$	**f.** $3 - 2x \geq 9$

20.1 Solving inequalities of the form $ax + b \gtreqless cx + d$

Let us now consider how to solve inequalities of the form $ax + b \gtreqless cx + d$ algebraically and represent the solutions geometrically on a number line.

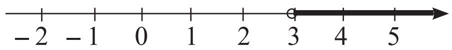
Example 1

Solve the inequality $3x - 2 > 2x + 1$ and represent the solutions on a number line.

When solving the inequality, all the terms with x should be carried to one side and the numerical terms should be carried to the other side (as in solving equations.)

$$3x - 2 > 2x + 1$$
$$3x - 2 + 2 > 2x + 1 + 2 \quad \text{(adding 2 to both sides)}$$
$$3x > 2x + 3$$
$$3x - 2x > 2x + 3 - 2x \quad \text{(subtracting } 2x \text{ from both sides.)}$$
$$\underline{\underline{x > 3}}$$

This is the solution of the inequality. In words we can say that the solutions of the inequality are all real numbers greater than 3. This can be represented on a number line as shown below.



To represent the fact that 3 is not a solution, a small un-shaded circle is drawn around the point denoting 3.

Example 2

Solve the inequality $5x + 3 \leq 3x + 1$ and represent the solutions of x on a number line.

$$5x + 3 \leq 3x + 1$$

$$5x + 3 - 3 \leq 3x + 1 - 3 \quad \text{(subtracting 3 from both sides)}$$

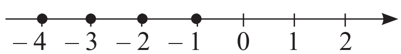
$$5x \leq 3x - 2$$

$$5x - 3x \leq 3x - 2 - 3x \quad \text{(subtracting } 3x \text{ from both sides)}$$

$$\frac{2x}{2} \leq \frac{-2}{2} \quad \text{(dividing both sides by 2)}$$

$$\underline{\underline{x \leq -1}}$$

Accordingly, the solutions of the inequality are all real numbers less than or equal to –1. The integral solutions of the inequality are all integers which are less than or equal to –1. That is –1, –2, –3 etc. This can be represented on a number line as shown below.



Note: If the integral solutions of the inequality are not asked for specifically, then all real numbers satisfying the inequality should be given as the solution.

Example 3

Solve the inequality $2x - 5 \geq 4x - 4$ and represent the solutions of x on a number line.

$$2x - 5 \geq 4x - 4$$

$$2x - 5 + 5 \geq 4x - 4 + 5 \quad \text{(adding 5 to both sides)}$$

$$2x \geq 4x + 1$$

$$2x - 4x \geq 4x + 1 - 4x \quad \text{(subtracting } 4x \text{ from both sides)}$$

$$-2x \geq 1$$

$$\frac{-2x}{-2} \leq -\frac{1}{2} \quad \text{(dividing both sides by } -2\text{)}$$

$$\underline{\underline{x \leq \frac{-1}{2}}}$$

Note: Remember that when you divide by a negative number the inequality sign changes. Consider how to solve this problem without having to divide by a negative number.

Exercise 20.1

1. Solve each of the inequalities given below. Represent the integral solutions of each inequality on a number line.

a. $3x - 4 > 2x$ **b.** $6x + 5 \geq 5x$

c. $2x - 9 \leq 5x$ **d.** $8 - 3x > x$

e. $5 - 2x \leq 3x$ **f.** $12 - x > 3x$

2. Solve each of the inequalities given below, and for each inequality, represent all the solutions on a number line.

a. $2x - 4 > x + 3$ **b.** $3x + 5 < x + 1$

c. $3x + 8 \geq 3 - 2x$ **d.** $5x + 7 \geq x - 5$

e. $3x - 8 \leq 5x + 2$ **f.** $2x + 3 \geq 5x - 6$

g. $x - 9 > 6x + 1$ **h.** $5x - 12 \leq 9x + 4$

i. $\frac{3x + 2}{2} > x + 3$ **j.** $2x - 5 \leq \frac{3x - 4}{-2}$

20.2 Solving problems using inequalities

Example 1

Eight tea packets of the same mass and three 1kg packets of sugar are in a shopping bag. The maximum mass the bag can hold is 5kg.

(i) Taking the mass of one tea packet as x grammes, write an inequality in terms of x.

(ii) Solve the inequality and find the maximum mass that a tea packet could be.

It is easier to work this problem out if all the masses are converted to grammes.

(i) Mass of a tea packet in grammes $= x$

Mass of 8 tea packets in grammes $= 8x$

Mass of sugar in grammes $= 3 \times 1000$

$= 3000$

Maximum mass the bag can hold in grammes $= 5 \times 1000$

$= 5000$

According to the information given, $8x + 3000 \leq 5000$

This is the required inequality.

(ii) $8x + 3000 \leq 5000$

$8x + 3000 - 3000 \leq 5000 - 3000$

$\frac{8x}{8} \leq \frac{2000}{8}$

$x \leq 250$

$\therefore$ the maximum mass of a tea packet is 250g.

Example 2

Sarath bought 5 exercise books and 3 pens while Kamani bought 3 exercise books and 11 pens. The amount spent by Sarath was greater than or equal to the amount spent by Kamani. Moreover, the price of a pen was Rs 10.

(i) Taking the price of an exercise book as Rs x, write an inequality in terms of x.

(ii) By solving the inequality, find the minimum price of an exercise book.

(i) Price of the exercise books Sarath bought = Rs $5x$

Amount Sarath spent = Rs $5x + 30$

Similarly, the amount Kamani spent = Rs $3x + 110$

According to the information given,

$$5x + 30 \geq 3x + 110$$

This is the required inequality.

(ii) $5x + 30 \geq 3x + 110$

$$5x + 30 - 30 \geq 3x + 110 - 30$$
$$5x \geq 3x + 80$$
$$5x - 3x \geq 3x + 80 - 3x$$
$$\frac{2x}{2} \geq \frac{80}{2}$$
$$x \geq 40$$

$\therefore$ the minimum price of an exercise book is Rs 40.

Exercise 20.2

1. 5 bags of cement of mass 50kg each and 30 wires of equal mass have been loaded into a small tractor. The maximum mass the tractor can carry is 700kg.

(i) Taking the mass of a wire as x, construct an inequality using the given information.
(ii) Find the maximum mass of a wire.

2. There are twelve small packets of biscuits and five 200g packets of biscuits in box A, while in box B there are four small packets of biscuits and nine 200g packets of biscuits. The mass of the biscuits in box A is less than or equal to the mass of the biscuits in box B.

(i) Taking the mass of a small packet of biscuits as x, write an inequality in terms of x using the given information.
(ii) Find the maximum mass of a small packet of biscuits.

3. There are trained and untrained employees in a workplace. The daily wage of a trained employee is Rs 1200. The amount spent on the daily wages of 5 trained employees and 7 untrained employees is greater than or equal to the amount spent on the daily wages of 7 trained employees and 4 untrained employees.

(i) Taking the daily wage of an untrained employee to be Rs x, construct an inequality using the information given above.
(ii) Solve the inequality and find the minimum daily wage of an untrained employee.

4. 5 packets of tea of equal mass and 3 kg of sugar are packed in a polythene bag. The mass that the polythene bag is greater than or equal to the mass of 25 packets of tea. Using this information, construct an inequality and find the mass of a tea packet that the bag can hold.

5. Square tiles of two sizes are used to tile two rooms. The area of the larger tile is $900cm^2$. To tile room A, 100 small tiles and 10 large tiles are needed, while to tile room B, 20 small tiles and 30 large tiles are needed. If the area of the floor of room B is greater than or equal to the area of the floor of room A, using an inequality, find the maximum length that a side of the smaller tile could be.

6. A large bucket of capacity 5 litres and a small bucket are used to fill a tank with water. The tank can be filled completely by using the large bucket 12 times and the small bucket 4 times (assuming that both buckets are filled to the brim). When the tank was filled using the large bucket 9 times and the small bucket 9 times, the tank did not overflow. Using an inequality, find the maximum capacity of the small bucket.

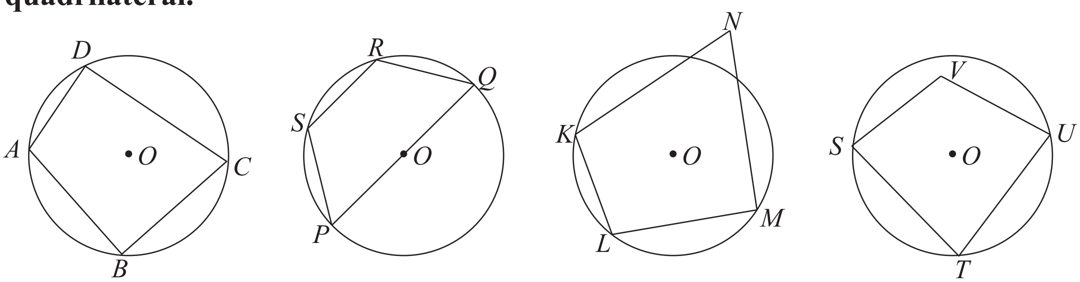
21 Cyclic Quadrilaterals

By studying this lesson you will be able to,

- identify cyclic quadrilaterals and identify the theorem that "the opposite angles of a cyclic quadrilateral are supplementary" and its converse,
- identify the theorem that "if one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral".

21.1 Cyclic quadrilaterals

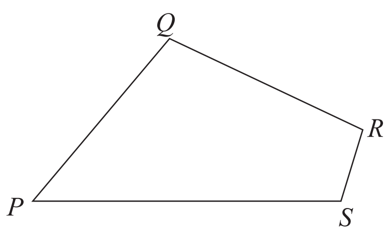
A quadrilateral which has all four vertices on the same circle is known as a **cyclic quadrilateral.**



(i) (ii) (iii) (iii)

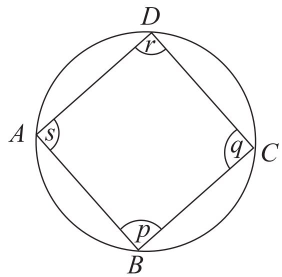
It is clear that the quadrilaterals in figures (i) and (ii) are **cyclic quadrilaterals** while the quadrilaterals in figures (iii) and (iv) are **not cyclic quadrilaterals**.

In a quadrilateral, the angle which is opposite a given angle is known as the "opposite angle". For example, in the quadrilateral shown below, $\hat{R}$ is the opposite angle of $\hat{P}$ and $\hat{S}$ is the opposite angle of $\hat{Q}$.



To understand the relationship between opposite angles in a cyclic quadrilateral let us do the following activity.

Activity



- Draw a cyclic quadrilateral as shown in the figure.
- Cut and separate out the angles in the cyclic quadrilateral.
- From the angles that were cut out, take the angles p and r and paste them on a piece of paper such that they are adjacent angles and see whether they form a pair of supplementary angles (that is, whether the sum of the magnitudes is 180°.) Do the same with the angles q and s.
- What is the conclusion you can draw regarding opposite angles of a cyclic quadrilateral?

You would have observed the $p + r = 180^\circ$ and $q + s = 180^\circ$. This relationship can be written as a theorem in the following form.

Theorem: The opposite angles of a cyclic quadrilateral are supplementary.

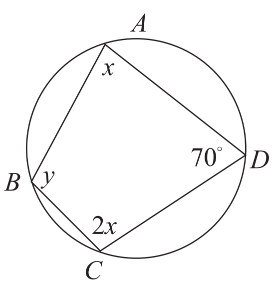
According to this theorem, in the above figure,

$$A\hat{B}C + C\hat{D}A = 180^\circ$$
$$D\hat{C}B + D\hat{A}B = 180^\circ$$

Let us now see how calculations are performed using the above theorem.

Example 1

Find the values of x and y in the cyclic quadrilateral $ABCD$ shown in the figure.



Since the opposite angles of a cyclic quadrilateral are supplementary,

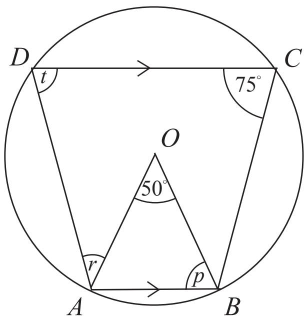
$$70^\circ + y = 180^\circ$$
$$\therefore \quad y = 180^\circ - 70^\circ$$
$$\underline{\underline{y = 110^\circ}}$$

Since the opposite angles of a cyclic quadrilateral are supplementary,

$$x + 2x = 180^\circ$$
$$3x = 180^\circ$$
$$\therefore\ \underline{\underline{x = 60^\circ}}$$

Example 2

O is the centre of the circle shown in the figure and *AB//DC*. Find the magnitudes of the angles denoted by symbols.



$O\hat{A}B = O\hat{B}A$ (*OA* and *OB* are equal as they are radii of the same circle)

$\therefore\ p + p + 50^\circ = 180^\circ$ (interior angles of the triangle *OAB*)

$$\therefore\ p = \frac{180^\circ - 50^\circ}{2}$$
$$= \underline{\underline{65^\circ}}$$

As the opposite angles of a cyclic quadrilateral add up to 180°,

$$75^\circ + D\hat{A}B = 180^\circ$$
$$D\hat{A}B = 180^\circ - 75^\circ$$
$$= 105^\circ$$

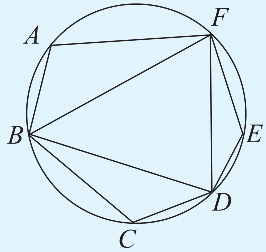
$$B\hat{A}O + O\hat{A}D = 105^\circ$$
$$\therefore\ 65^\circ + r = 105^\circ$$
$$r = 105^\circ - 65^\circ$$
$$r = \underline{\underline{40^\circ}}$$

As allied angles add up to 180°,

$$t + 105^\circ = 180^\circ$$
$$\therefore\ t = 180^\circ - 105^\circ$$
$$t = \underline{\underline{75^\circ}}$$

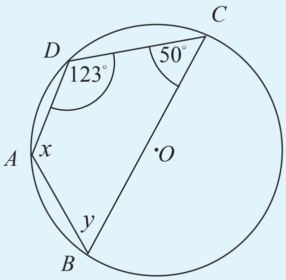
Exercise 21.1

1. (i) Write down all the cyclic quadrilaterals in the figure.
(ii) For each of the cyclic quadrilaterals written above, write down the pairs of opposite angles.

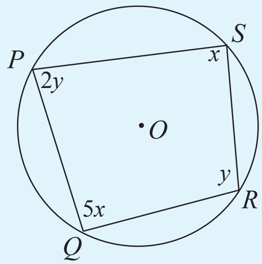


2. Find the magnitude of each of the angle denoted by a symbol, based on the information in the figure. O denotes the centre of each circle.

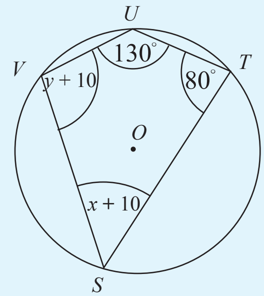
(i)



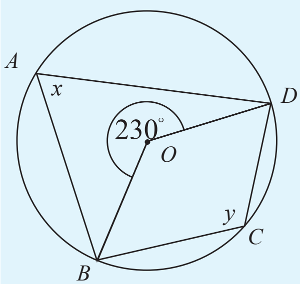
(ii)



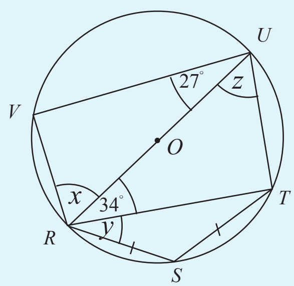
(iii)



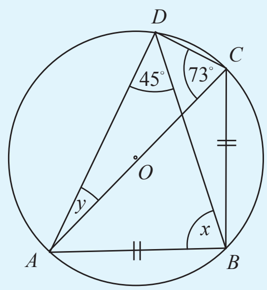
(iv)



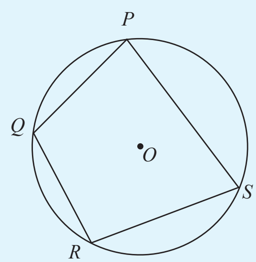
(v)



(vi)



3. O is the centre of the circle shown in the figure.



a. If $\hat{P} = 60^\circ$ and $\hat{S} = 125^\circ$, then find the magnitudes of $\hat{R}$ and $\hat{Q}$.

b. If $\hat{P} : \hat{R} = 2 : 3$, then find the magnitudes of $\hat{P}$ and $\hat{R}$.

c. If $\hat{Q} - \hat{S} = 120^\circ$, then find the magnitudes of $\hat{S}$ and $\hat{Q}$.

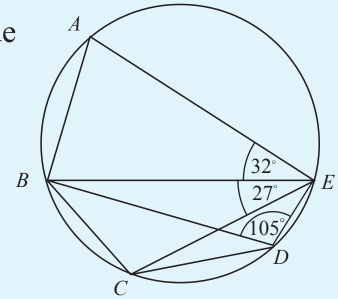
d. If $2\hat{P} = \hat{R}$, then find the magnitude of $\hat{P}$.

e. If $\hat{P} = 2x + y$, $\hat{Q} = x + y$, $\hat{R} = 60^\circ$ and $\hat{S} = 90^\circ$ then find the values of x and y.

4. The points denoted by A, B, C, D, E and F lie on the circumference of the circle with centre O. Find the value of $F\hat{A}B + B\hat{C}D + D\hat{E}F$.

5. Using the information in the figure, find the magnitude of each of the angles given below.

a. $B\hat{A}E$ **b.** $C\hat{B}A$ **c.** $C\hat{B}E$



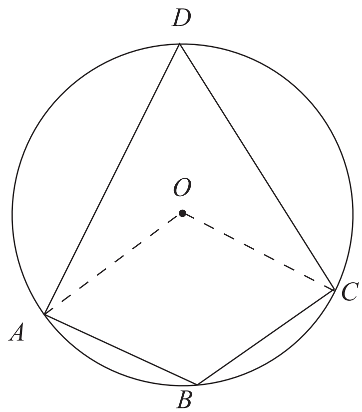
Now let us see how to prove the theorem, "the opposite angles of a cyclic quadrilateral are supplementary."

Data: $ABCD$ is a cyclic quadrilateral with its vertices on the circle with centre O.

To be proved: $A\hat{B}C + A\hat{D}C = 180°$ and

$D\hat{A}B + D\hat{C}B = 180°$.

Construction: Join OA and OC.



Proof: $A\hat{O}C = 2\,A\hat{D}C$

(the angle subtended at the centre is twice the angle subtended at the circumference)

$A\hat{O}C$ (reflex) $= 2\,A\hat{B}C$ (the angle subtended at the centre is twice the angle subtended at the circumference)

$\therefore\ A\hat{O}C + A\hat{O}C$ (reflex) $= 2\,A\hat{D}C + 2\,A\hat{B}C$

But, $A\hat{O}C + A\hat{O}C$ (reflex) $= 360°$ (angles around a point)

$\therefore\ 2\,A\hat{D}C + 2\,A\hat{B}C = 360°$

Therefore, $A\hat{D}C + A\hat{B}C = 180°$

We can join OB and OD, and similarly prove that $D\hat{A}B + D\hat{C}B = 180°$.

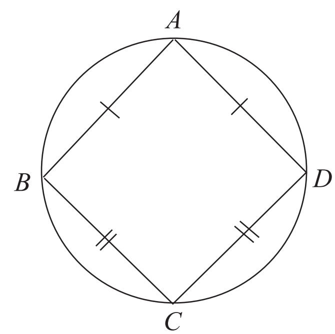
$\therefore$ The opposite angles of a cyclic quadrilateral are supplementary.

The converse of this theorem is also true. That is, if the sum of the opposite angles of a quadrilateral is 180°, then its vertices lie on a circle. We can write this as a theorem as below.

Theorem: If the opposite angles of a quadrilateral are supplementary, then the vertices of the quadrilateral are on the circle.

Now let us consider how riders are proved using the above theorem.

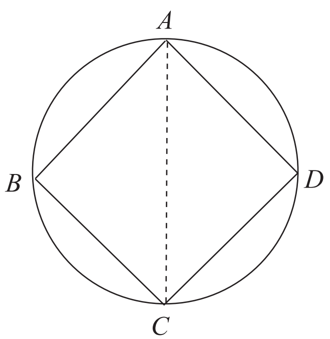
Example 1



In the cyclic quadrilateral shown in the figure, $AB = AD$ and $CB = CD$.

(i) Show that $\Delta ABC \equiv \Delta ACD$

(ii) Deduce that AC is a diameter.



(i) When we consider the triangles ABC and ADC,

$AB = AD$ (given)

$BC = DC$ (given)

AC is the common side

$\therefore \quad \Delta ABC \equiv \Delta ACD \quad$ (SSS)

(ii) $A\hat{B}C = A\hat{D}C$ (corresponding angles of congruent triangles are equal)

But, $A\hat{B}C + A\hat{D}C = 180^\circ$ (opposite angles of a cyclic quadrilateral are supplementary)

$\therefore \quad A\hat{B}C + A\hat{B}C = 180^\circ$

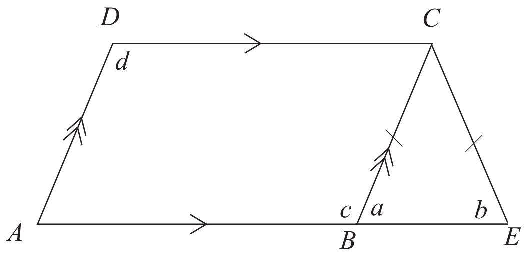
$\therefore 2\, A\hat{B}C = 180^\circ$

$\therefore A\hat{B}C = 90^\circ$

$\therefore$ AC is a diameter (angle in a semicircle is 90°).

Example 2

In the parallelogram *ABCD*, *AB* is produced to *E* such that $CB = CE$. Show that *AECD* is a cyclic quadrilateral.



$a = b$ (since $CE = CB$)

$c = 180^\circ - a$ (angles on a straight line)

$c = 180^\circ - b$ (since $a = b$) ——— ①

$c = d$ (opposite angles of the parallelogram *ABCD*) ——— ②

From ① and ②,

$d = 180^\circ - b$

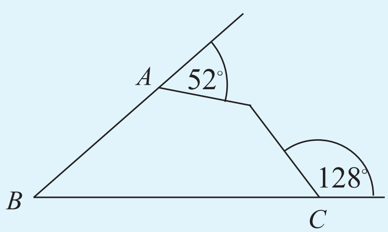
$\therefore\ b + d = 180^\circ$

As the opposite angles of the quadrilateral *AECD* add up to 180°, this is a cyclic quadrilateral.

Exercise 21.2

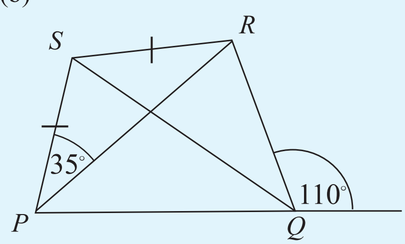
1. State with reasons whether each of the following quadrilaterals is a cyclic quadrilateral or not.

(a)



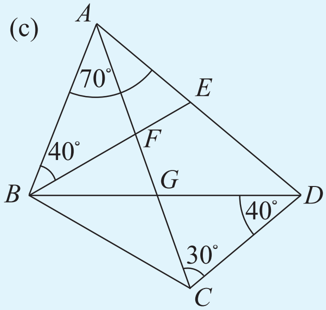
Quadrilateral *ABCD*

(b)



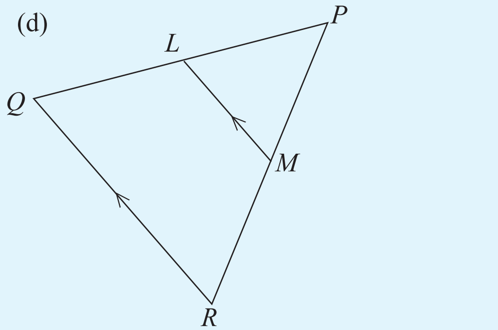
Quadrilateral *PQRS*

(c)



Quadrilateral *FGDE*

(d)



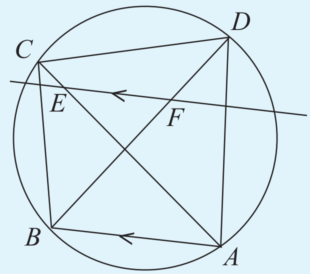
Quadrilateral *QRML* if $PQ = PR$

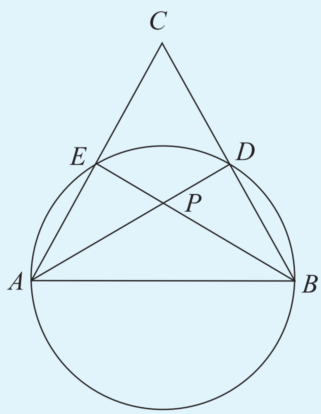
2. Show that *PQRS* is a cyclic quadrilateral if $\hat{P} = \hat{Q}$ and $\hat{R} = \hat{S}$.

3. In the cyclic quadrilateral *ABCD, AC* is joined. Show that $B\hat{A}C = A\hat{D}C - A\hat{C}B$.

4. Show that *A,B,C* and *D* are points on the same circle if $A\hat{B}D + A\hat{D}B = D\hat{C}B$ in the quadrilateral *ABCD*.

5. Using the information in the figure, prove that *CDFE* is a cyclic quadrilateral.





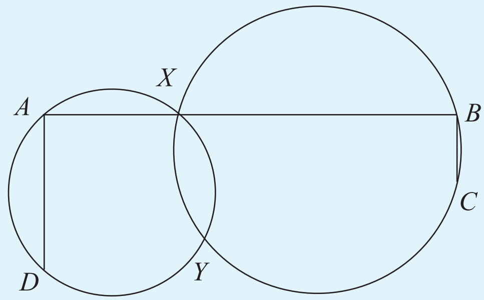
6. If *AB* is a diameter of the circle in the figure, show that $A\hat{P}B = C\hat{A}B + A\hat{B}C$.

7. In the triangle PQR, PQ is produced to S and PR is produced to T. The bisectors of $S\hat{Q}R$ and $Q\hat{R}T$ meet at X and the bisectors of $P\hat{Q}R$ and $P\hat{R}Q$ meet at Y.

(i) Show that $QXRY$ is a cyclic quadrilateral with XY as a diameter of the corresponding circle.

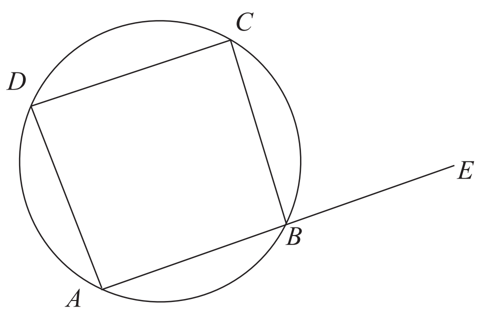
(ii) Find the magnitude of $Q\hat{X}R$ if $Q\hat{P}R = 40^\circ$.

8. The two circles in the figure intersect at X and Y. A straight line drawn through X meets the two circles at A and B. If D and C are marked on the circles such that $AD//BC$, show that DYC is a straight line.

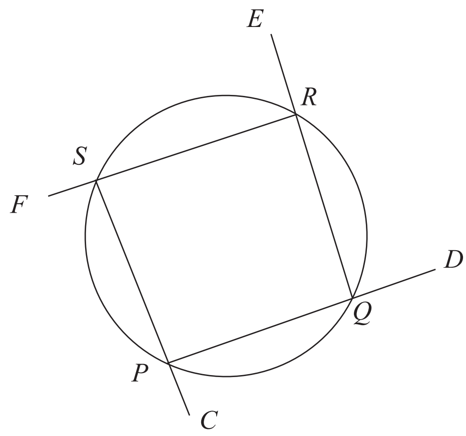


21.3 The relationship between the exterior angle and the interior opposite angle

In the cyclic quadrilateral $ABCD$ shown in the figure, AB is produced to E.



Then, $C\hat{B}E$ is an exterior angle of the cyclic quadrilateral. The corresponding interior opposite angle is $A\hat{D}C$.

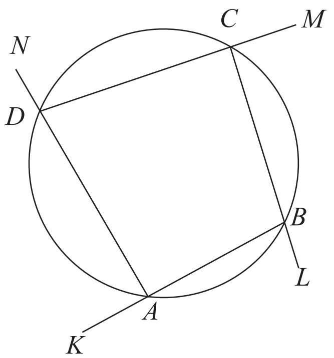


The following table has been filled by considering the cyclic quadrilateral *PQRS* in the figure.

Produced side	**Exterior angle**	**Interior opposite angle**
PQ	$D\hat{Q}R$	$P\hat{S}R$
QR	$E\hat{R}S$	$Q\hat{P}S$
RS	$F\hat{S}P$	$P\hat{Q}R$
SP	$Q\hat{P}C$	$Q\hat{R}S$

The relationship between an exterior angle and the interior opposite angle of a cyclic quadrilateral is given in the theorem below.

Theorem: If one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.



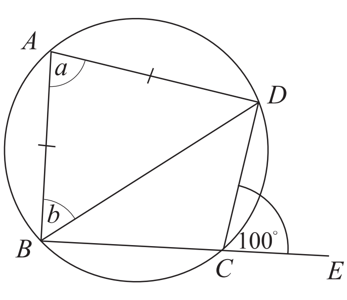
For the above figure, the following are true according to the theorem.

$$D\hat{A}K = B\hat{C}D$$
$$A\hat{B}L = C\hat{D}A$$
$$B\hat{C}M = B\hat{A}D$$
$$C\hat{D}N = A\hat{B}C$$

Let us consider why this theorem is true. Let us for example consider why the angles $D\hat{A}B$ and $B\hat{C}M$ are equal. Since $ABCD$ is a cyclic quadrilateral, $D\hat{A}B + B\hat{C}D = 180^\circ$. Furthermore, since DCM is a straight line, $D\hat{A}B + B\hat{C}D = B\hat{C}D + B\hat{C}M$. Cancelling $B\hat{C}D$ from both sides, we obtain $D\hat{A}B = B\hat{C}M$.

Example 1

Find the values of a and b based on the information in the given figure.

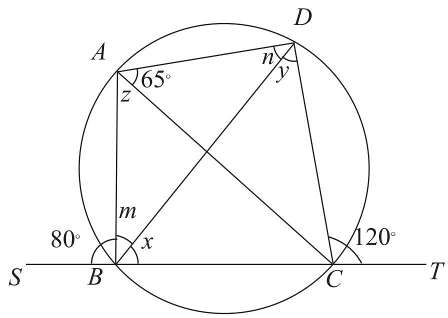


As the exterior angle of the cyclic quadrilateral is equal to the interior opposite angle,
$a = \underline{\underline{100^\circ}}$.

$A\hat{D}B = b$ (since $AB = AD$)
$a + b + b = 180^\circ$ (interior angles of a triangle)
$100^\circ + 2b = 180^\circ$
$b = \underline{\underline{40^\circ}}$

Example 2

Find the values of x, y, z, m and n based on the information in the given figure.



$x = 65^\circ$ (angles in the same segment)

As an exterior angle of a cyclic quadrilateral is equal to the interior opposite angle,

$$B\hat{A}D = D\hat{C}T$$
$$B\hat{A}D = 120^\circ$$
$$z + 65^\circ = 120^\circ$$
$$z = 55^\circ$$
$z = y$ (angles in the same segment)
$$\therefore \; y = \underline{\underline{55^\circ}}$$

As an exterior angle of a cyclic quadrilateral is equal to the interior opposite angle,

$$A\hat{D}C = A\hat{B}S = 80^\circ$$
$$\therefore \; n + y = 80^\circ$$
$$n + 55^\circ = 80^\circ$$
$$n = 80^\circ - 55^\circ$$
$$\therefore \; n = \underline{\underline{25^\circ}}$$

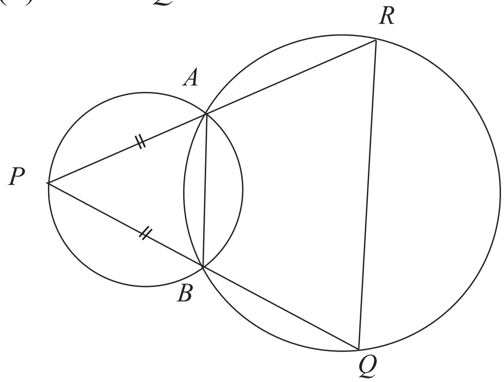
$80^\circ + m + x = 180^\circ$ (angles on a straight line)
$$\therefore \; 80^\circ + m + 65^\circ = 180^\circ$$
$$m = 180^\circ - 145^\circ$$
$$m = \underline{\underline{35^\circ}}$$

Example 3

The two circles in the given figure intersect at A and B. Moreover, $PA = PB$. If $A\hat{P}B = 70^\circ$,

(i) find the magnitude of $A\hat{R}Q$.

(ii) Is $AB//RQ$?



(i) In the triangle APB,

$P\hat{A}B = P\hat{B}A$ (since $PA = PB$)

$\therefore \; P\hat{A}B = P\hat{B}A = \dfrac{180^\circ - 70^\circ}{2} = 55^\circ$

Moreover, $A\hat{B}P = A\hat{R}Q$ (exterior angle of the cyclic quadrilateral $ABQR$ = the interior opposite angle)

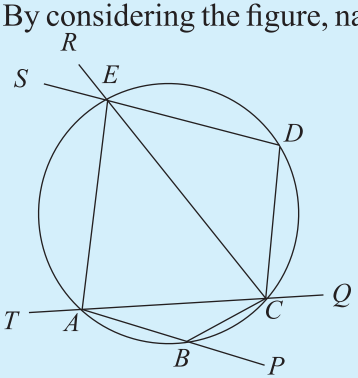
$\therefore \; A\hat{R}Q = \underline{\underline{55^\circ}}$

(ii) $P\hat{A}B = A\hat{R}Q = 55^\circ$.

$\therefore \; AB//RQ$. (since corresponding angles are equal)

Exercise 21.3

1. By considering the figure, name an angle equal to each of the angles given below.

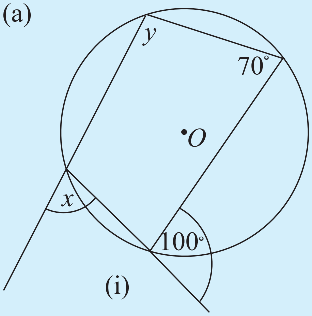


(i) $C\hat{B}P$ (ii) $D\hat{C}Q$ (iii) $R\hat{E}A$

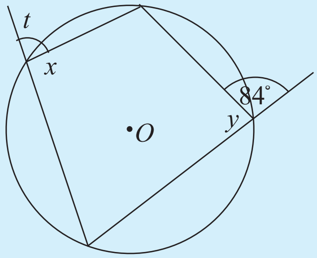
(iv) $S\hat{E}A$ (v) $E\hat{A}T$

2. In each of the circles given below, the centre is O. Find the magnitude of each of the angles denoted by an algebraic symbols.

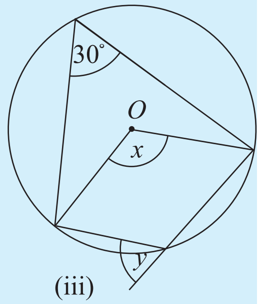
(a)



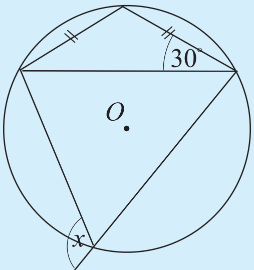
(i)



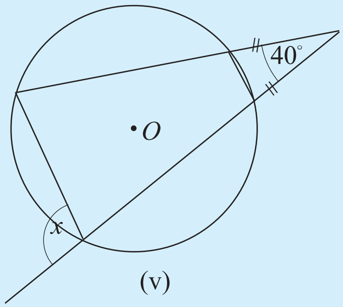
(ii)



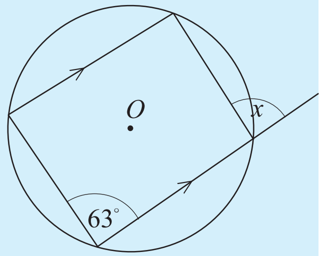
(iii)



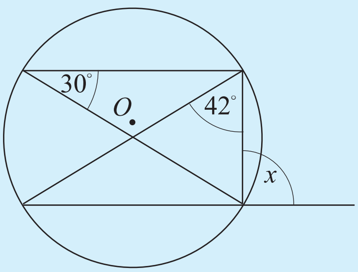
(iv)



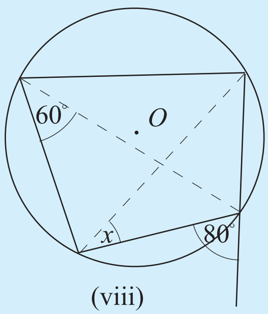
(v)



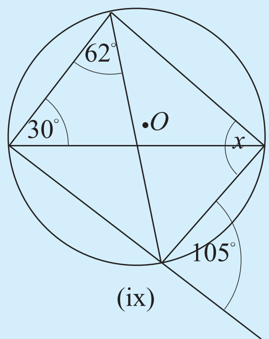
(vi)



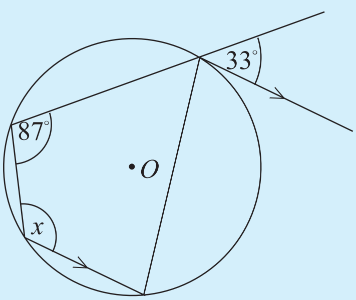
(vii)



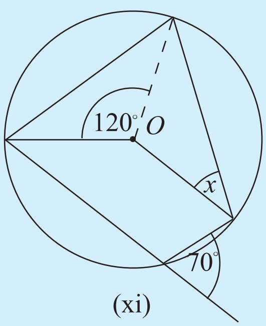
(viii)



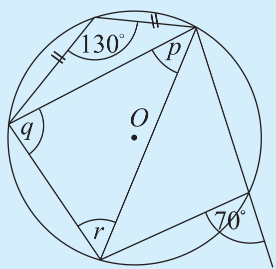
(ix)



(x)

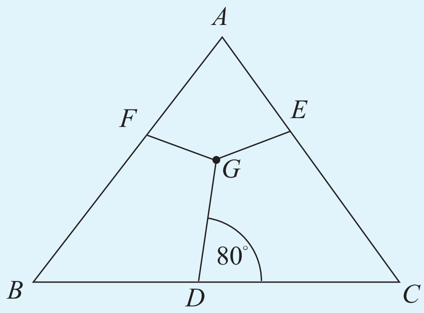


(xi)



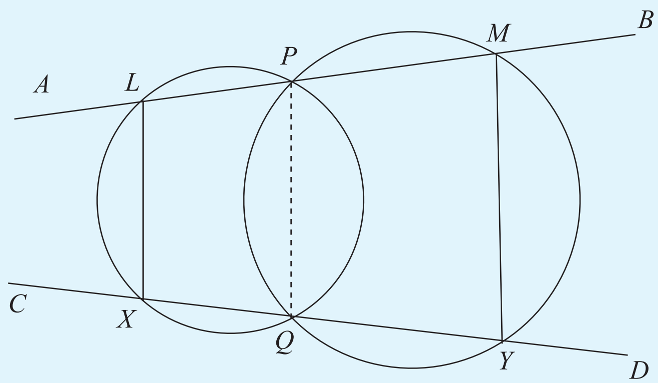
(xii)

3. The points D, E and F are on the sides BC, CA and AB respectively of the triangle ABC such that $BDGF$ and $DCEG$ are cyclic quadrilaterals. Furthermore, $G\hat{D}C = 80^\circ$.



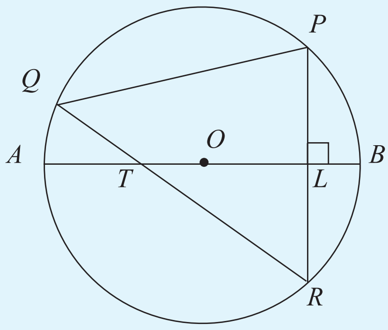
(i) Find the magnitudes of $A\hat{F}G$ and $A\hat{E}G$.
(ii) Show that $AFGE$ is a cyclic quadrilateral.

4. The circles given in the figure intersect at P and Q. The straight lines APB and CQD meet the circles at L, P, M and X, Q, Y respectively.



(i) Find the magnitude of $B\hat{M}Y$ if $A\hat{L}X = 105^\circ$.
(ii) Show that LX and MY are parallel.

5. The centre of the given circle is O. The diameter AB and the chord PR intersect perpendicularly at L. The line segments QR and AB intersects at T.



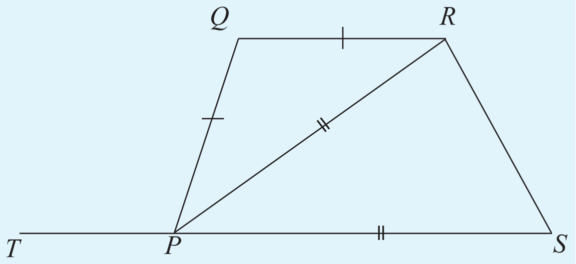
a. If $Q\hat{T}A = x$, write in terms of x,
   (i) the magnitude of $L\hat{R}T$,
   (ii) the magnitude of $O\hat{P}Q$.

b. Show that $QTOP$ is a cyclic quadrilateral.

6. In the figure, $PQ = QR$ and $PR = PS$. If $P\hat{R}S = 2\,Q\hat{R}P$, show that

(i) $PSRQ$ is a cyclic quadrilateral,

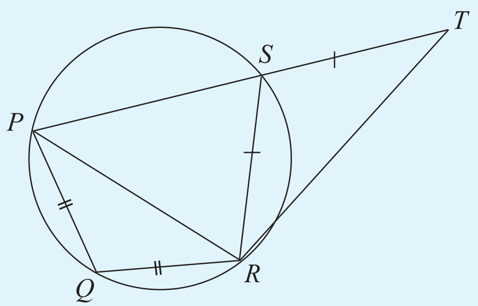
(ii) $Q\hat{P}T : P\hat{R}S = 3 : 2$.



7. $PQ = QR$ in the cyclic quadrilateral $PQRS$. Moreover, PS is produced to T such that $RS = ST$. If $S\hat{R}T = 32^\circ$,

(i) find the magnitude of $Q\hat{R}P$,

(ii) show that QS and RT are parallel.



22 Tangents

By studying this lesson you will be able to,

- identify the tangent which is drawn through a point on a circle and its characteristics,
- identify the tangents drawn to a circle from an external point and their characteristics,
- identify the angles in the alternate segment and solve related problems.

22.1 Tangents

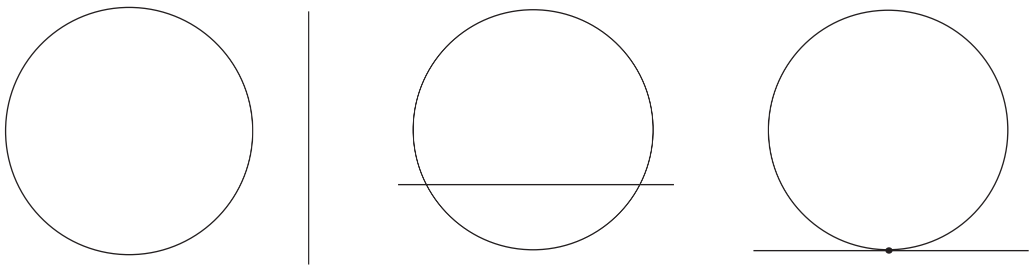


Figure (i) Figure (ii) Figure (iii)

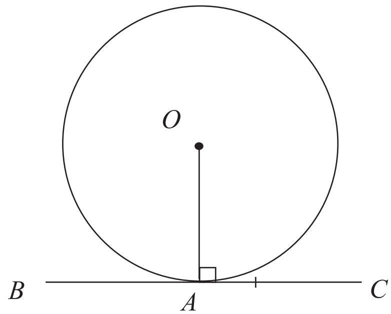
In Figure (i), the circle and the straight line have no points in common. Therefore the straight line is situated external to the circle.

In Figure (ii), the circle and the straight line intersect at two points. Therefore the circle and the straight line have two points in common. The straight line is known as a secant of the circle.

In Figure (iii), the circle and the straight line have one point in common. Here the straight line touches the circle and therefore the straight line is known as a "**tangent**" of the circle. The point which is common to the tangent and the circle is known as the **tangential point**.

Line drawn through a point on a circle, perpendicular to the radius

To learn about the line drawn through a point on a circle which is perpendicular to the radius, consider the facts given below.



O is the centre of the circle in the figure. The radius drawn through the point A which is on the circle is OA. The line drawn perpendicular to the radius OA through A is BC. It is clear that $O\hat{A}C = 90^\circ$ and that BC is a tangent to the circle.
That is,
the line BC drawn perpendicular to the radius OA through the point A is a tangent to the circle.
This result can be written as a theorem as follows.

Theorem: The straight line drawn through a point on a circle and perpendicular to the radius through the point of contact, is a tangent to the circle.

Furthermore the converse of the above theorem is also true.

That is,

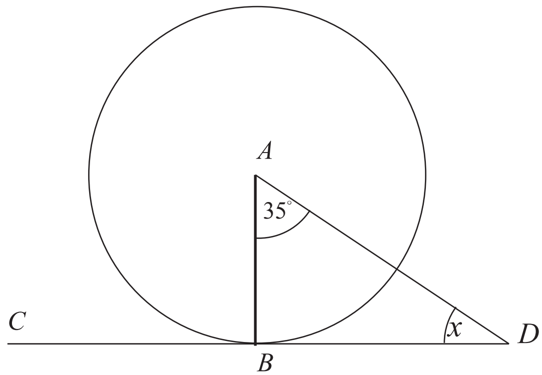
if a tangent is drawn to the circle through a particular point, then the radius that passes through that point and the said tangent are perpendicular to each other.

This result too can be written as a theorem as follows.

Converse of the theorem: The tangent through a point on a circle is perpendicular to the radius drawn to the point of contact.

Example 1

The tangent drawn to the circle with centre A through the point B is CD. If $B\hat{A}D = 35^{\circ}$, then find the value of x.

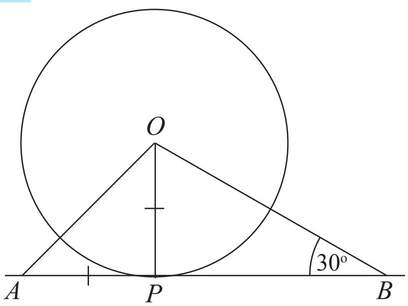


$A\hat{B}D = 90^{\circ}$ (the tangent through a point on a circle is perpendicular to the radius drawn to the point of contact)

As the sum of the interior angles of a triangle is equal to 180°,

$$35^{\circ} + 90^{\circ} + x = 180^{\circ}$$
$$x = 180^{\circ} - 35^{\circ} - 90^{\circ}$$
$$\underline{\underline{x = 55^{\circ}}}$$

Example 2



In the figure, the tangent drawn to the circle with centre O through P is AB. If $OP = AP$ and $O\hat{B}P = 30^{\circ}$, find the magnitude of $A\hat{O}B$.

$O\hat{P}A = 90^{\circ}$ (the tangent through a point on a circle is perpendicular to the radius drawn to the point of contact)

$OP = AP$ (given)

$\therefore P\hat{O}A = P\hat{A}O$ (in an isosceles triangle, the angles opposite equal sides are equal)

In the triangle APO,

$P\hat{A}O + P\hat{O}A + O\hat{P}A = 180^\circ$ (the sum of the interior angles of a triangle is 180°)

$\therefore \; P\hat{A}O + P\hat{O}A + 90^\circ = 180^\circ$

$P\hat{A}O + P\hat{O}A = 180^\circ - 90^\circ$

$P\hat{A}O + P\hat{O}A = 90^\circ$

$\therefore 2\,P\hat{A}O = 90^\circ$ (since $P\hat{A}O = P\hat{O}A$)

$$P\hat{A}O = \frac{90^\circ}{2}$$

$= 45^\circ$

In the triangle AOB,

$A\hat{O}B + P\hat{A}O + P\hat{B}O = 180^\circ$ (the sum of the interior angles of a triangle is 180°)

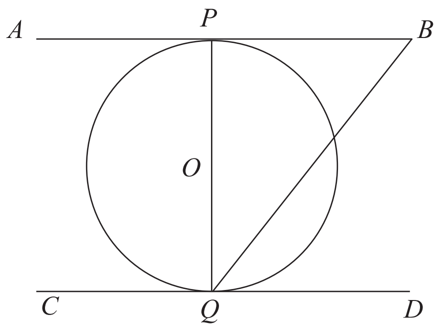
$A\hat{O}B + 45^\circ + 30^\circ = 180^\circ$

$A\hat{O}B + 75^\circ = 180^\circ$

$A\hat{O}B = 180^\circ - 75^\circ$

$= \underline{\underline{105^\circ}}$

Example 3



PQ is a diameter of the circle with centre O. The tangents drawn to the circle through P and Q, are AB and CD respectively. Show that $P\hat{B}Q = B\hat{Q}D$.

Since the tangent drawn to the circle through a point on the circle is perpendicular to the radius drawn to the tangential point,

$Q\hat{P}B = 90^\circ$ and

$P\hat{Q}D = 90^\circ$.

$\therefore \; Q\hat{P}B + P\hat{Q}D = 90^\circ + 90^\circ$

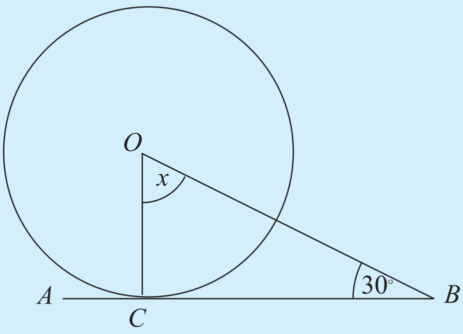
$= 180^\circ$

$\therefore \; AB \,//\, CD$ (allied angles)

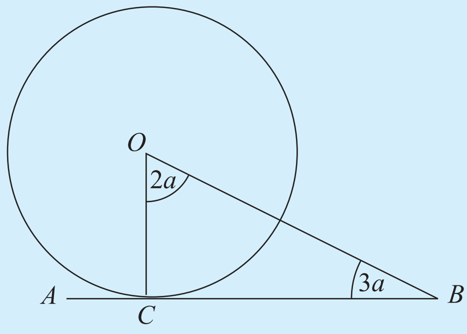
$\therefore \; P\hat{B}Q = B\hat{Q}D$ ($AB \,//\, CD$ and alternate angles)

Exercise 22.1

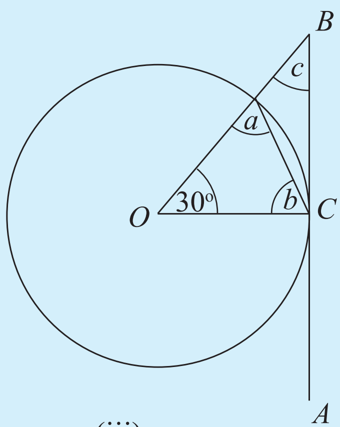
1. In each circle given below, the centre is O and AB is the tangent drawn to the circle through the point C. Find the value of each algebraic symbol based on the data in the figure.



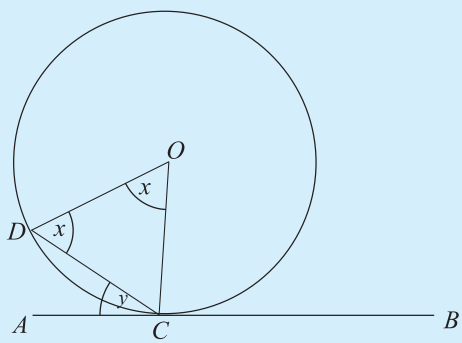
(i)



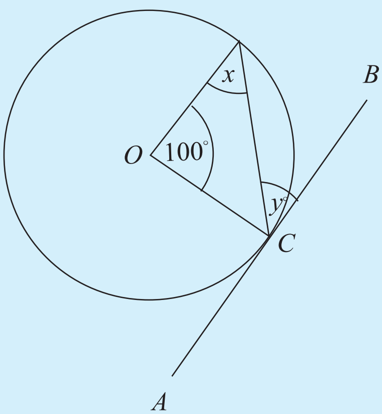
(ii)



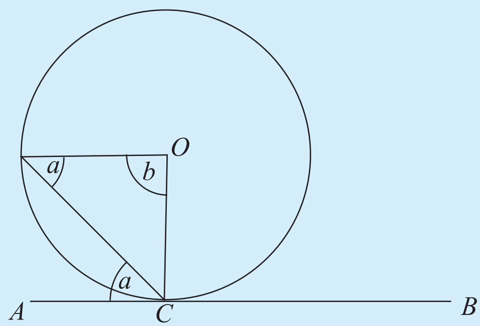
(iii)



(iv)

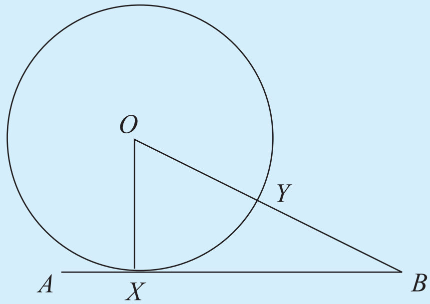


(v)

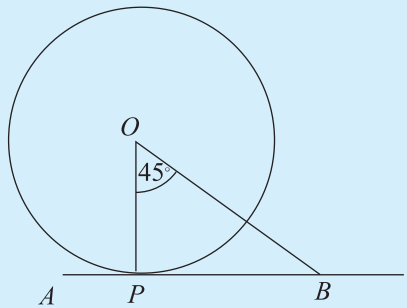


(vi)

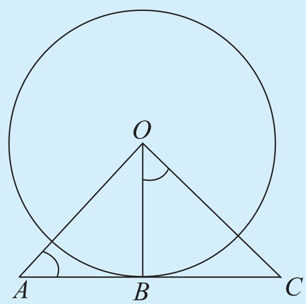
2. In the circle with centre O in the figure, AB is the tangent to the circle through the point X. If the radius of the circle is 6 cm and $YB = 4$ cm, find the length of XB.



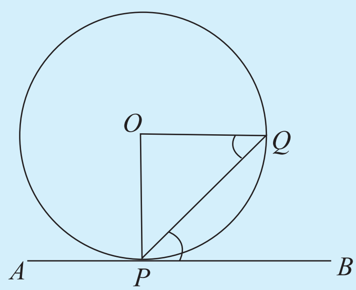
3. The tangent drawn to the circle with centre O through the point P is AB. Find the radius of the circle if $B\hat{O}P = 45^{\circ}$ and $PB = 6$ cm.



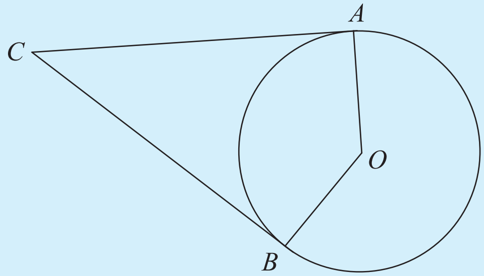
4. The tangent drawn to the circle with centre O through the point B is AC. If $O\hat{A}B = B\hat{O}C$ then show that $A\hat{O}B = B\hat{C}O$.



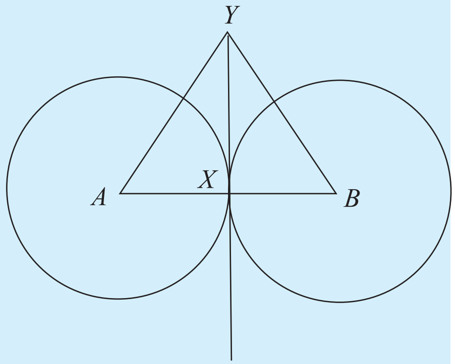
5. The tangent drawn to the circle with centre O though the point P is AB. Q is on the circle such that $O\hat{Q}P = Q\hat{P}B$. Show that OQ and PO are perpendicular to each other.



6. The tangents drawn to the circle with centre O through the points A and B intersect at point C. Show that $AOBC$ is a cyclic quadrilateral.

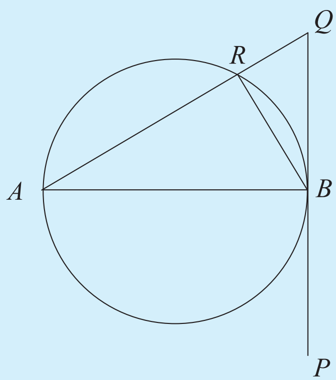


7. Two circles with equal radii and with centres A and B are shown in the figure. Y is situated such that $AY = YB$. Show that YX is a common tangent to the two circles.



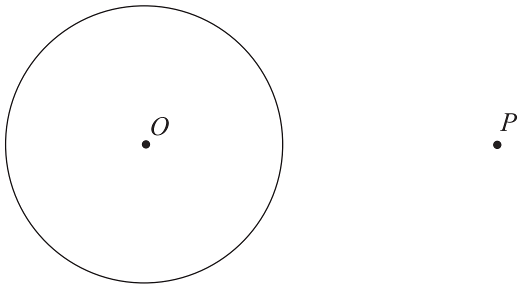
8. In the given figure, AB is a diameter of the circle and PQ touches the circle at the point B. Prove that,

(i) $Q\hat{R}B = 90^{\circ}$ and
(ii) $A\hat{B}R = R\hat{Q}B$

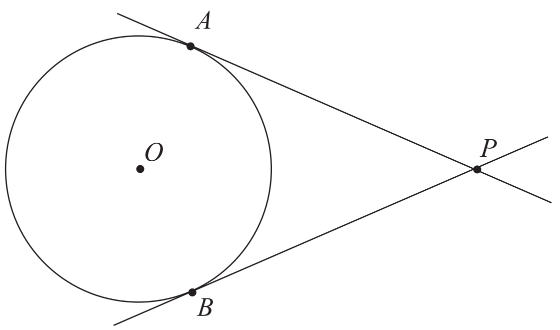


22.2 Tangents drawn to a circle from an external point

Let us consider a point *P* which is external to the circle with centre *O*.

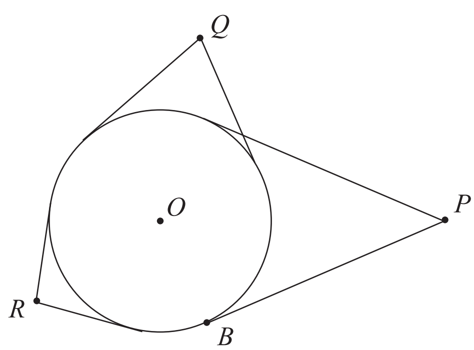


Two tangents to the circle that pass through the point *P* can be drawn. These two tangents are shown in the following figure.

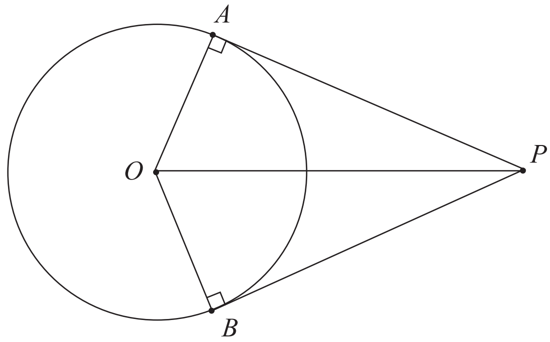


These two tangents are known as the tangents drawn to the circle from the external point *P*.

Understand that there are two tangents that can be drawn to a circle from an external point *P* irrespective of where the point is located. In the figure below are three points *P*, *Q* and *R* through which two tangents each are drawn.



Now let us consider the characteristics of the two tangents drawn from an external point to a circle.
Let us mark A and B as the tangential points and OA and OB as the respective radii. Let us also draw the straight line segment OP.



As learnt in section 22.1 above, tangents are perpendicular to the radii drawn through the tangential points. This is indicated in the above figure.

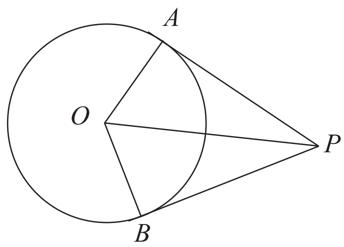
Observing the triangles OAP and OBP, we might guess that they are congruent, based on symmetry. In fact they are congruent. This can easily be proved. Let us see how this is proved. First observe that they are both right angled triangles. Therefore if we show that the hypotenuse and one side of one triangle are equal to the hypotenuse and corresponding side of the other triangle, we can conclude that they are congruent under the RHS (Hyp.S) case. In both triangles, the hypotenuse is the common side OP. Furthermore, since OA and OB are both radii, they are equal in length. Accordingly, these two triangles are congruent under the RHS case. As they are congruent,

(i) $AP = BP$; that is, the tangents are equal in length.
(ii) $A\hat{P}O = B\hat{P}O$; that is, the angle between the two tangents is bisected by OP.
(iii) $A\hat{O}P = B\hat{O}P$; that is, both tangents subtend equal angles at the centre of the circle.

The above considered facts are expressed as a theorem as follows.

Theorem: If two tangents are drawn to a circle from an external point, then,
(i) the two tangents are equal in length.
(ii) the angle between the tangents is bisected by the straight line joining the external point to the centre.
(iii) the tangents subtend equal angles at the centre.

Let us now consider how this theorem is proved formally.



Data : The tangents drawn from the external point P through the points A and B on the circle with centre O are AP and BP respectively.

To prove :

(i) $AP = BP$

(ii) $A\hat{P}O = B\hat{P}O$

(iii) $P\hat{O}A = P\hat{O}B$

Proof : $O\hat{A}P = O\hat{B}P = 90^{\circ}$ (tangents are perpendicular to the radii)

$\therefore$ POA and POB are right angled triangles.

Now, in the triangles POA and POB,

$OA = OB$ (radii of the same circle)

OP is the common side

$\therefore$ $\Delta POA \equiv \Delta POB$ (Hyp.S)

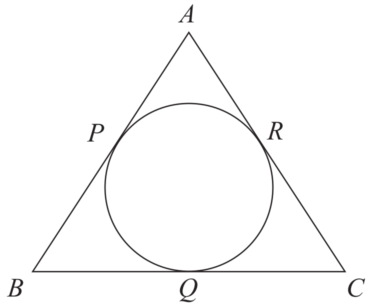
In congruent triangles, corresponding sides and corresponding angles are equal.

$\therefore$ (i) $AP = BP$

$\therefore$ (ii) $A\hat{P}O = B\hat{P}O$

$\therefore$ (iii) $P\hat{O}A = P\hat{O}B$

Example 1



As shown in the figure, the circle touches the triangle ABC at the points P, Q and R. If $AB = 11$cm and $CR = 4$ cm, find the perimeter of the triangle ABC.

Tangents drawn from an external point to a circle are equal in length.

$$\therefore \; AP = AR$$
$$BP = BQ$$
$$CR = CQ$$

$\therefore$ The perimeter of $\Delta\ ABC$ $= AB + BC + CA$

$= 11 + (BQ + QC) + (CR + RA)$

$= 11 + (BP + CR) + (CR + AP)$

$= 11 + (BP + 4) + (4 + AP)$

$= 19 + (BP + AP)$

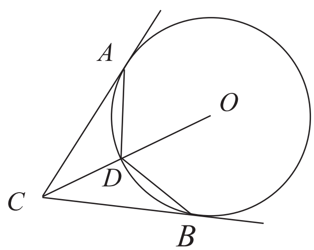
$= 19 + AB$

$= 19 + 11$

$= 30$

$\therefore$ The perimeter of the triangle ABC is 30 cm.

Example 2



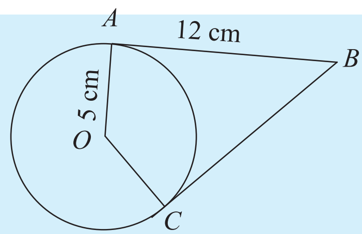
As shown in the figure, the two tangents drawn from the external point C to the circle with centre O touches the circle at A and B. The line drawn from the centre O to C intersects the circle at D.
Show that $AD = BD$.

We can obtain the required result by proving that the two triangles ACD and BCD are congruent.

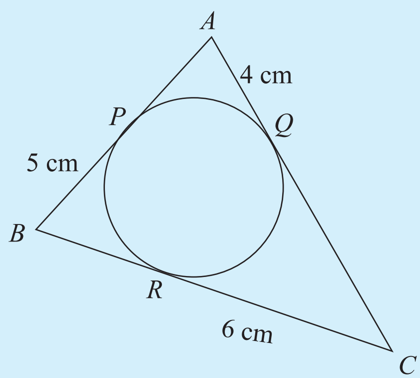
In the triangles ACD and BCD,
$AC = BC$ (tangents drawn to a circle from an external point are equal in length)
$A\hat{C}O = B\hat{C}O$ (the angle between the tangents is bisected by the straight line joining the external point to the centre)
CD is the common side
$\therefore \Delta ACD \equiv \Delta BCD$ (SAS)
$\therefore \underline{\underline{AD = BD}}$ (corresponding sides of two congruent triangles)

Exercise 22.2

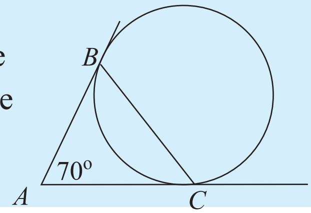
1. The tangents through the points A and C on the circle with centre O in the figure, meet at B. If the radius of the circle is 5 cm and AB = 12 cm, then find the perimeter of the quadrilateral $ABCO$.



2. The tangents drawn to the circle through the points P, Q and R on the circle are AB, AC and BC respectively. Find the perimeter of the triangle ABC if RC = 6 cm, BP = 5 cm and AQ = 4 cm.

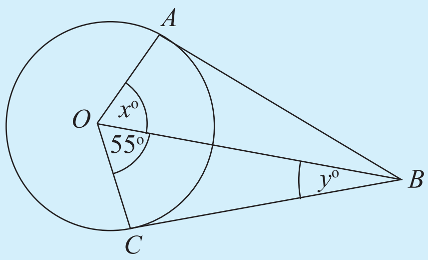


3. The tangents drawn to the circle in the figure through the points B and C meet at A. If $B\hat{A}C = 70^\circ$, find the magnitude of $A\hat{B}C$.

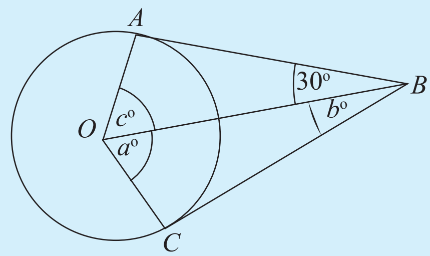


4. The centre of each of the circles shown below is O. The tangents drawn to the circle through the points A and C meet at B. Based on the given data, find the values represented by the algebraic symbols.

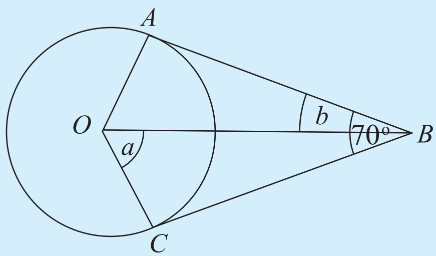
(i)



(ii)

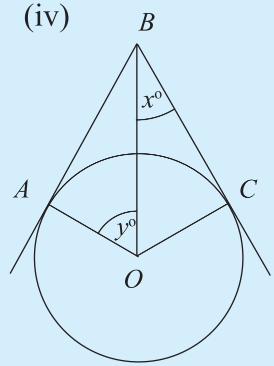


(iii)



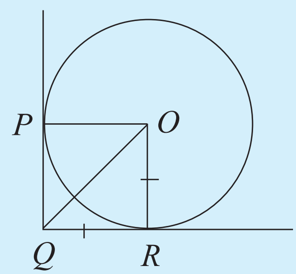
$A\hat{B}C = 70^\circ$

(iv)

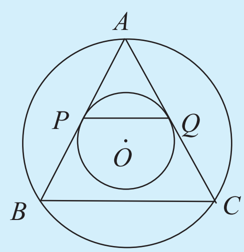


$A\hat{O}C = 110^\circ$

5. O is the centre of the circle shown in the figure. The tangents drawn to the circle through the points P and R meet at Q. If $QR = OR$, show that $PQRO$ is a square.

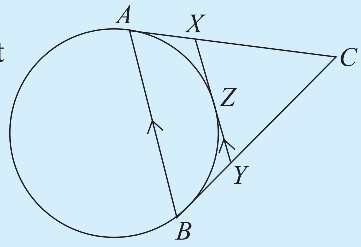


6. A, B and C are points on the larger circle with centre O shown in the figure. AB and AC touch the smaller circle at the points P and Q. Show that,



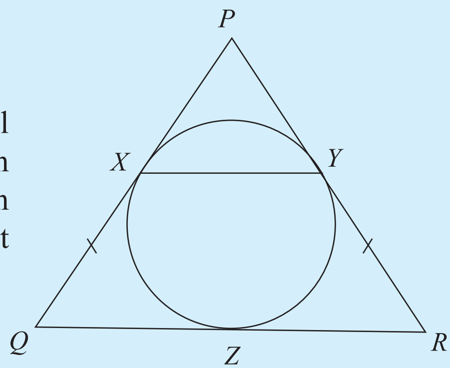
(i) APQ is an isosceles triangle,
(ii) $BC \,//\, PQ$.

7. According to the information in the figure, show that $XC = CY$. Also, AC and BC are the tangents drawn to the given circle at A and B respectively.



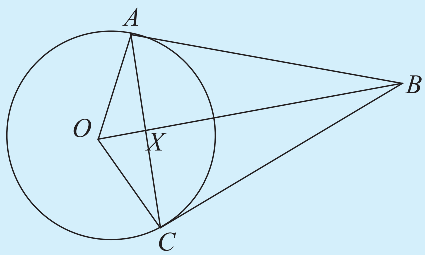
8. The tangents drawn from the external point P to the circle in the figure, touch the circle at X and Y. The line QR which touches the circle at Z is drawn such that $XQ = YR$. Show that,

(i) $PR = PQ$,
(ii) $QR = XQ + YR$,
(iii) $XY \parallel QR$.



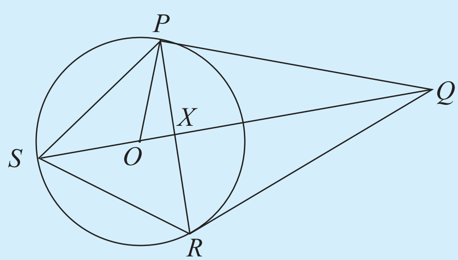
9. The tangents drawn through the points A and C on the circle with centre O shown in the figure meet at the point B. Show that,

(i) $OAX\Delta \equiv OCX\Delta$,
(ii) OB is the perpendicular bisector of AC,
(iii) $A\hat{O}C = 2\,A\hat{C}B$.



10. The tangents drawn from the external point Q to the circle with centre O shown in the figure are PQ and QR. QO produced meets the circle at S. Show that,

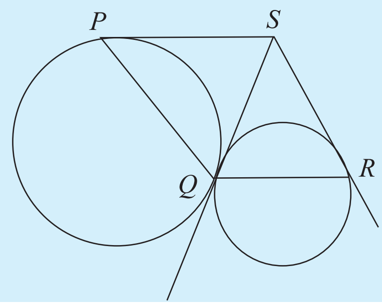
(i) $\Delta PQS \equiv \Delta QRS$
(ii) $2\,O\hat{P}X = P\hat{Q}R$.



11. As shown in the figure, Q is a point on both circles and QS is a tangent to both circles. The other tangents drawn from S to the two circles touch the circles at P and R. Show that,

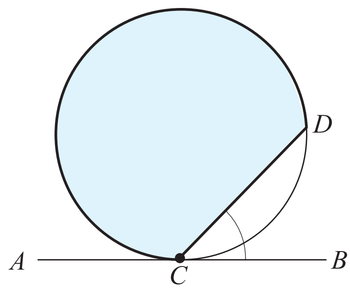
(i) $PS = SR$,

(ii) $P\hat{Q}R = S\hat{P}Q + S\hat{R}Q$.



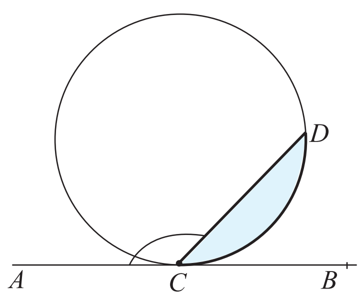
22.3 Angles in the alternate segment

First let us see what an alternate segment is. To do this let us consider the following figure.



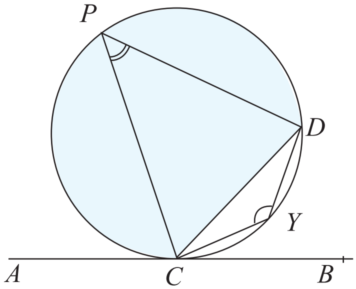
As shown in the figure, the straight line AB touches the circle at the point C. In the figure, CD is a chord. The circle is divided into two segments by the chord CD. One segment is the portion which is shaded light blue. The other segment is the smaller portion which is not shaded. There are two angles formed by the chord CD meeting the tangent AB. One angle is $A\hat{C}D$. The other is $B\hat{C}D$ The alternate segment corresponding to the angle $B\hat{C}D$ is the portion shaded light blue. Similarly, the alternate segment corresponding to the angle $A\hat{C}D$ is the portion of the circle which is not shaded.

In the figure given below, the alternate segment corresponding to $A\hat{C}D$ is shaded light blue.



Theorems related to angles in the alternate segment

Consider the figure given below. $C\hat{P}D$ is in the larger segment which is shaded light blue. Therefore the angle $C\hat{P}D$ is in the alternate segment of $D\hat{C}B$. Similarly $C\hat{Y}D$ is in the alternate segment of $A\hat{C}D$.

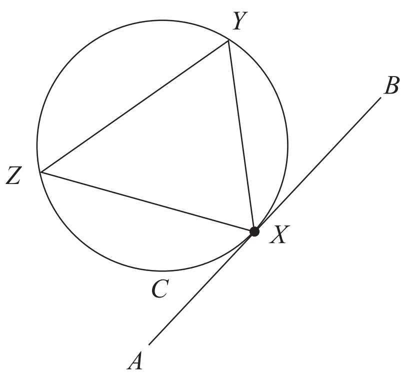


There is a special relationship involving tangents of a circle. For the above figure, it means that the angles $D\hat{C}B$ and $C\hat{P}D$ are equal, and also that the angles $A\hat{C}D$ and $C\hat{Y}D$ are equal. In other words, "The angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle". This is a very important result. Therefore let us write it as a theorem and remember it.

Theorem: The angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle**.**

To check the validity of the above theorem let us do some activities.

Activity 1

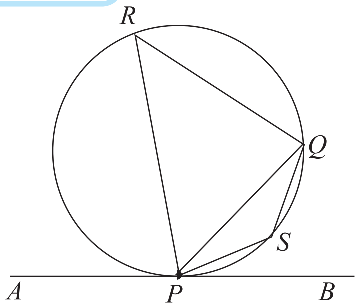


- Draw a circle and mark a point X on it.
- Draw a line touching the circle at X (draw a radius from X and draw a line

perpendicular to it.) Name it AB.

- Mark two other points on the circle and name them Y and Z.
- As shown in the figure, join the points X, Y and Z.
- Using the protractor, measure the magnitude of $B\hat{X}Y$ and also the magnitude of $X\hat{Z}Y$, which is the corresponding angle in the alternate segment, and check whether they are equal.
- Similarly, find the magnitude of $A\hat{X}Z$ and the magnitude of $X\hat{Y}Z$, which is the corresponding angle in the alternate segment, and check whether they are equal

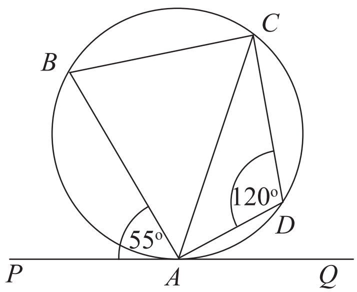
Activity 2



- Draw a circle and mark a point on it and name it P. Draw a line touching the circle at P (this can be done by drawing a radius from P and drawing a line perpendicular to it through P). Name it AB.
- Draw a chord from P and name it PQ.
- Mark two points on the circle on opposite sides of the chord and name them R and S.
- Draw the line segments QR, QS, PS and PR.
- Using the protractor, measure the magnitude of $B\hat{P}Q$ and the magnitude of $P\hat{R}Q$, which is the angle in the alternate segment, and check whether they are equal.
- Similarly measure the magnitude of $A\hat{P}Q$ and the magnitude of $P\hat{S}Q$, which is the angle in the alternate segment, and check whether they are equal.

From the above activities you would have understood that the angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.

Example 1



In the above figure, the line PQ touches the circle at A. Furthermore, B, C and D are also on the circle. $P\hat{A}B = 55^\circ$ and $A\hat{D}C = 120^\circ$. Find the magnitude of $B\hat{A}C$.

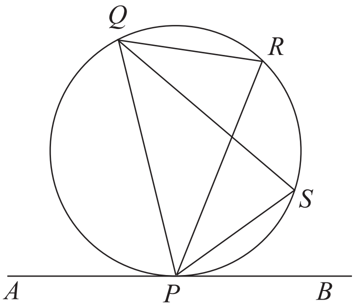
Initially let us find the magnitude of the angle $P\hat{A}C$.

$P\hat{A}C = A\hat{D}C$ (the angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle)

$$P\hat{A}B + B\hat{A}C = 120^\circ$$
$$55^\circ + B\hat{A}C = 120^\circ$$
$$B\hat{A}C = 120^\circ - 55^\circ$$
$$= \underline{\underline{65^\circ}}$$

Example 2

The straight line AB touches the circle at point P, Q and R are points on the circle. The bisector of the angle $P\hat{Q}R$ meets the circle at S. Show that PS is the bisector of the angle $B\hat{P}R$.



$B\hat{P}S = P\hat{Q}S$ (the angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle)

$R\hat{P}S = R\hat{Q}S$ (angles in the same segment are equal)

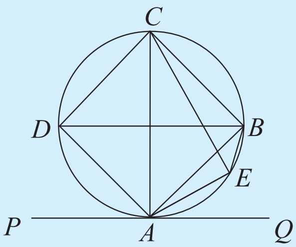
$P\hat{Q}S = R\hat{Q}S$ (data, because QS bisects the angle $P\hat{Q}R$)

$\therefore\ B\hat{P}S = R\hat{P}S$

$\therefore\ PS,$ is the angle bisector of $B\hat{P}R$

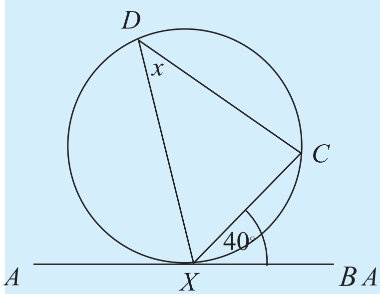
Exercise 22.3

1. PQ is a tangent to the circle through the point A. The points B, C, D and E lie on the circle.

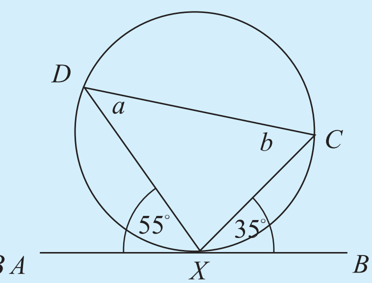


Angle between the tangent and the chord	Angle in the alternate segment
$B\hat{A}Q$	....
$P\hat{A}B$	....
$P\hat{A}D$	....
$E\hat{A}Q$	....
....	$D\hat{B}A$
....	$D\hat{C}A$

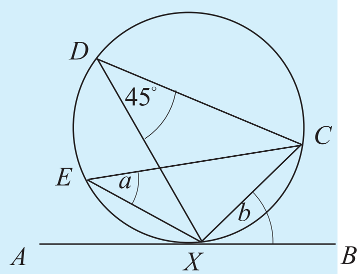
2. In each of the following figures, AB is the tangent to the circle drawn through the point X. Find the values represented by the algebraic symbols.



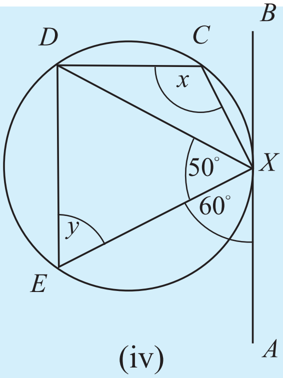
(i)



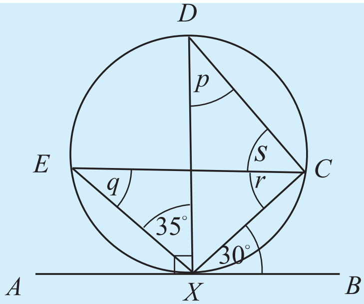
(ii)



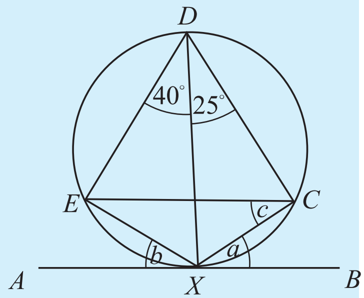
(iii)



(iv)



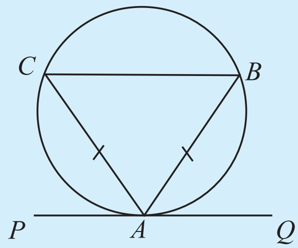
(v)



(vi)

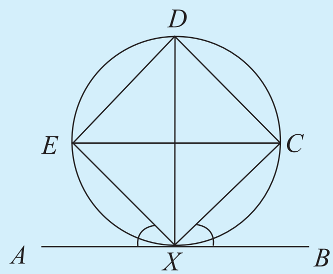
3. *PQ* is the tangent to the circle through *A*. If $AC = AB$ show that,

(i) $C\hat{A}P = B\hat{A}Q$,
(ii) $PQ \parallel CB$.



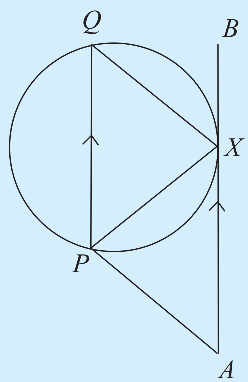
4. *AB* is a tangent drawn to the circle through *X*. The points *C* and *E* are on the circle such that $B\hat{X}C = A\hat{X}E$. *D* is another point on the circle. Show that,

(i) *XD* is the bisector of $E\hat{D}C$,
(ii) $EX = CX$,
(iii) $AB \parallel EC$.



5. The straight line *AB* touches the circle *at X*. The chord *PQ* is drawn such that $PQ \parallel AB$. Prove that,

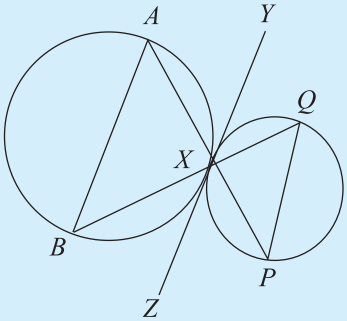
(i) $B\hat{X}Q = A\hat{X}P$,
(ii) *AXQP* is a parallelogram if $PX = PA$.



6. Two circles touch each other externally at X. YZ is the common tangent. AB is a chord of one circle. AX produced and BX produced meet the other circle at P and Q respectively. Show that,

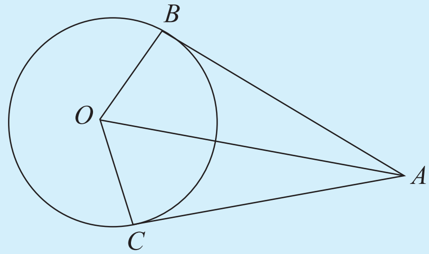
(i) $B\hat{X}Z = X\hat{P}Q$,

(ii) $AB \parallel PQ$.

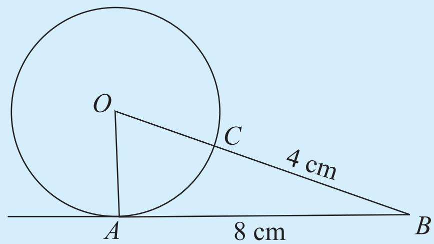


Miscellaneous Exercise

1. The tangents drawn to a circle with centre O from an external point A meet the circle at B and C. If the radius of the circle is 5 cm and OA is 13 cm, find the area of the quadrilateral $OBAC$.



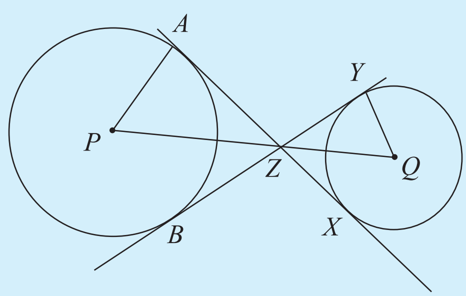
2. AB is the tangent drawn to the circle with centre O through the point A. OB intersects the circle at C. If $CB = 4$ cm and $AB = 8$ cm, find the radius of the circle.



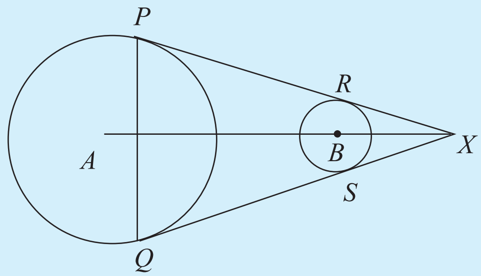
3. The centres of the two circles shown in the figure are P and Q. The two tangents drawn to the larger circle through A and B touch the smaller circle at X and Y respectively. Moreover, these two tangents intersect at Z. Show that,

(i) $AX = BY$,

(ii) $A\hat{P}Z = Y\hat{Q}Z$.



4. As shown in the figure, the tangents PX and QX touch the circles at P, R, Q and S. The centres of the circles are A and B. Show that,



(i) $PR = QS$,
(ii) $PQ \,//\, RS$,
(iii) A, B and X are on the same straight line.

23 Constructions

By studying this lesson you will be able to,

- do constructions related to straight lines and angles,
- construct circles related to triangles,
- construct tangents to circles.

23.1 Constructions related to straight lines and angles

Let us learn some constructions which will be required in the constructions that will be studied later on. We use only a pair of compasses and a straight edge to do constructions.

1. Construction of the perpendicular bisector of a straight line segment

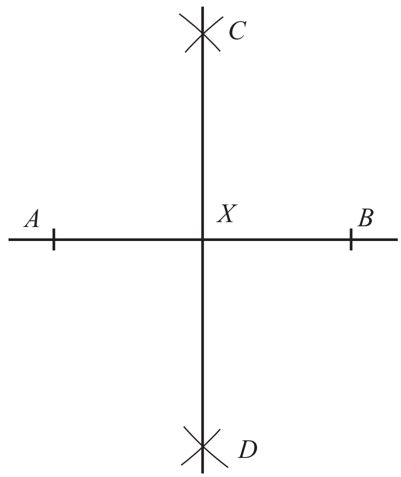
The perpendicular bisector of a straight line segment is the line drawn perpendicular to the straight line segment through its midpoint.
Let us consider a straight line segment AB.

Step 1: Take a length of a little more than half the length of AB onto the pair of compasses. Taking A as the centre, draw two arcs above and below the straight line segment.

Step 2: Using the same length (i.e., without altering the pair of compasses) and taking B as the centre, draw another two arcs such that they intersect the two arcs drawn earlier.

Step 3: Name the two points of intersection as C and D. Draw a straight line segment joining C to D.

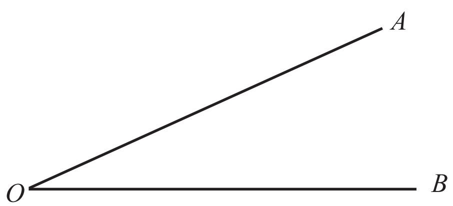
Step 4: Name the point where this straight line segment intersects AB as X.



CD is the perpendicular bisector of the straight line segment *AB*. Using a protractor, measure the magnitudes of the angles $A\hat{X}C$, $B\hat{X}C$, $A\hat{X}D$ and $B\hat{X}D$, and using a cm/mm scale, measure the lengths of *AX* and *BX*. Thereby establish the fact that *CD* is the perpendicular bisector of *AB*.

2. Construction of an angle bisector

Consider the angle $A\hat{O}B$

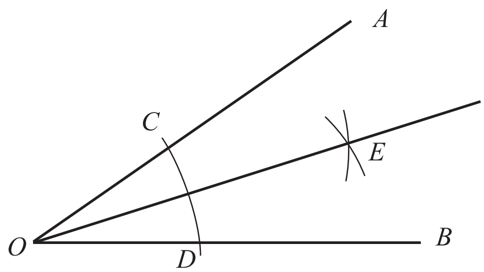


Step 1: Take a length which is less than *OA* and *OB* onto the pair of compasses. Taking *O* as the centre, draw an arc such that it intersects both *OA* and *OB*.

Step 2: Name the two points of intersection of the arc with *OA* and *OB* as *C* and *D*.

Step 3: Taking a suitable length onto the pair of compasses, and taking *C* and *D* as the centres, draw two arcs which intersect each other. Name the point of intersection of the two arcs as *E*.

Step 4: Join *O* and *E*.



OE is the angle bisector of $A\hat{O}B$. Establish this fact by measuring the magnitudes of the angles $A\hat{O}E$ and $B\hat{O}E$.

3. Construction of a perpendicular to a line through a given point on the line.

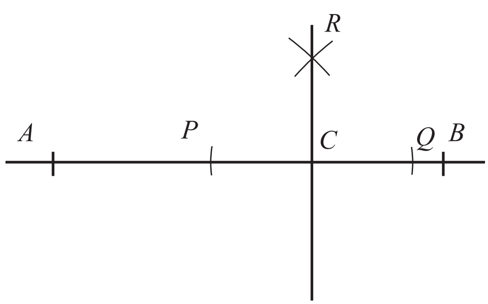
Let us assume that we want to draw a perpendicular to AB through the point C which is on AB.

Step 1: Take a suitable length onto the pair of compasses and taking C as the centre, draw two arcs that intersect the straight line segment AB on the two sides of C.

Step 2: Name the two points of intersection as P and Q.

Step 3: Taking P and Q as the centres and using a fixed radius, draw two arcs above or below the line AB such that they intersect each other.

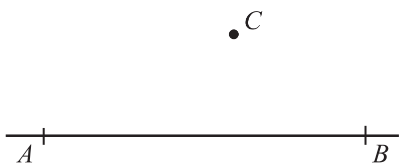
Step 4: Name the point of intersection of the two arcs as R, and join CR with a straight line.



CR is the perpendicular drawn to AB through C. Measure the angles $A\hat{C}R$ and $B\hat{C}R$ and establish this fact.

4. Construction of a perpendicular to a straight line segment from an external point

Let AB be a straight line segment and C an external point.

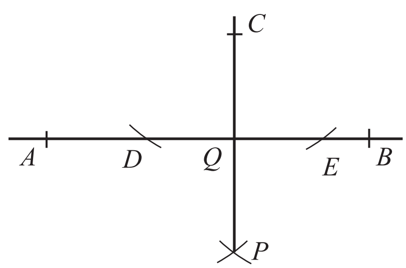


Step 1: Take a length which is a little more than the distance from C to AB onto the pair of compasses. Taking C as the centre, draw two arcs which intersect AB.

Step 2: Name the two points of intersection as D and E.

Step 3: Taking the same radius (or another suitable one), draw two intersecting arcs taking D and E as the centres, on the side of AB opposite to that on which C lies.

Step 4: Name the point of intersection of the two arcs as P and join CP. Name the point of intersection of CP and AB as Q.



CP is the perpendicular drawn to AB from the point C. This can be established by measuring the magnitudes of the angles $C\hat{Q}A$ and $C\hat{Q}B$ using the protractor.

Exercise 23.1

1. Construct the perpendicular bisector of the straight line segment AB where $AB = 5.2$ cm.

2. Construct an angle of 90° and construct its bisector.

3. Construct the triangle ABC where $AB = 6$ cm, $A\hat{B}C = 60°$ and $BC = 5$ cm. Construct the perpendicular bisector of AB.

4. (i) Construct the triangle PQR where $PQ = 7$cm, $QR = 6.5$ cm and $PR = 5$ cm.
   (ii) Construct the bisectors of $Q\hat{P}R$ and $P\hat{Q}R$.

5. (i) Draw the straight line segment XY of length 5.5 cm.
   (ii) Construct a perpendicular to XY through X.
   (iii) Mark a point 4 cm from X on the perpendicular and name it Z. Join YZ. Construct a perpendicular from X to YZ.

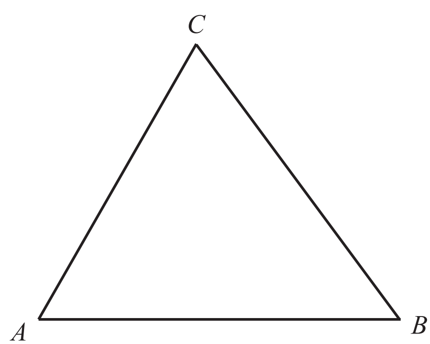
6. (i) Construct an equilateral triangle ABC of side length 6 cm.
   (ii) Construct a perpendicular from each vertex of traingle ABC to the opposite side.

23.2 Construction of circles related to triangles

You have learnt earlier how to construct triangles using a straight edge and a pair of compasses when lengths of the sides of the triangle and magnitudes of the angles are given. Now let us learn how to construct circles related to triangles in three cases, using only a straight edge and a pair of compasses.

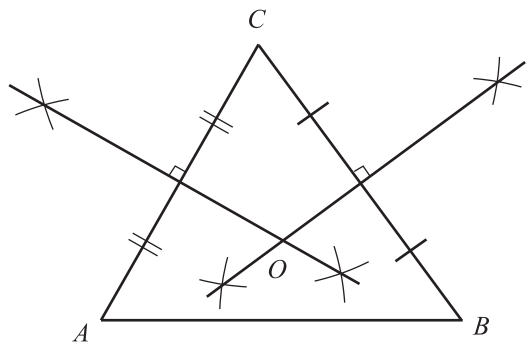
Construction of the circumcircle (circumscribed circle) of a triangle

Draw a triangle and name it ABC.

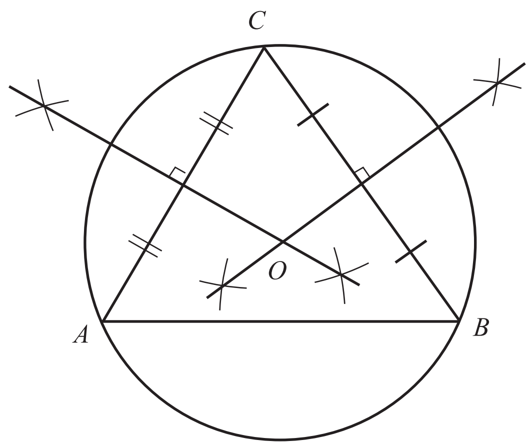


Step 1: Using the pair of compasses, draw the perpendicular bisectors of any two of the three sides *AB, BC* and *CA* of the triangle *ABC*.

Step 2: Name the point of intersection of these perpendicular bisectors as *O*.



Step 3: Taking *O* as the centre and the distance from any vertex of the triangle to *O* as the radius, draw a circle.



Observe that the constructed circle passes through all the vertices of the triangle *ABC*. This circle is known as the circumcircle of the triangle *ABC*. The centre of this circle is known as the circumcentre of the triangle.

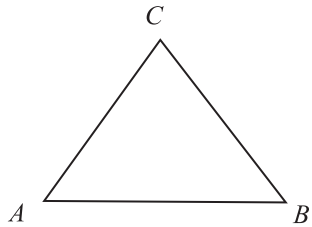
Draw a right angled triangle and an obtuse angled triangle and construct the circumcircle of each triangle.

Using the above constructions, fill in the table given below.

Triangle	Location of the circumcentre		
	Inside the triangle	On a side of the triangle	Outside the triangle
Acute angled triangle	✓	×	×
Right angled triangle			
Obtuse angled triangle			

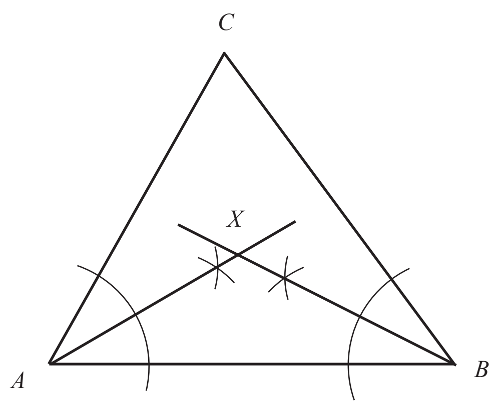
Construction of the incircle (inscribed circle) of a triangle

Draw a triangle and name it *ABC*.

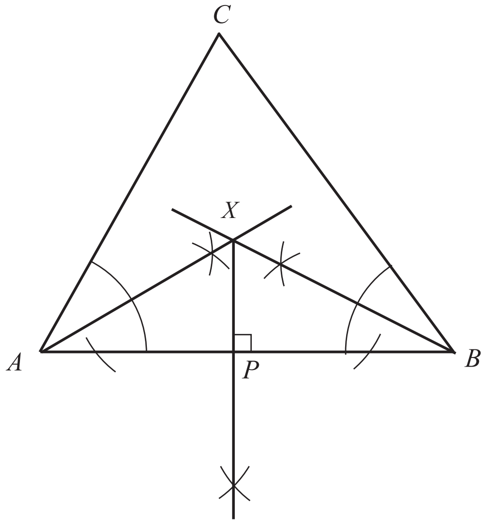


Step 1: Using the pair of compasses construct the angle bisectors of any two interior angles of the triangle *ABC*.

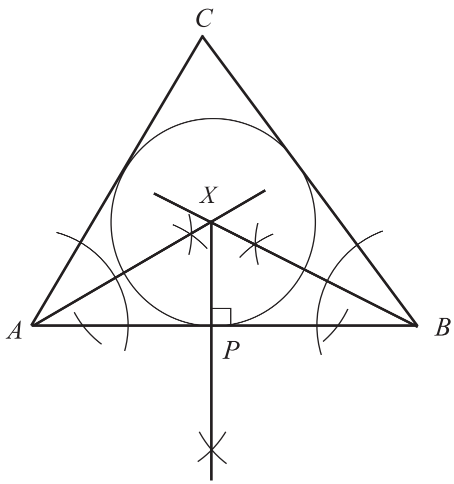
Step 2: Name the point at which the angle bisectors meet as *X*.



Step 3: Construct a perpendicular from *X* to any one of the three sides of the triangle. Name the foot of that perpendicular as *P*.



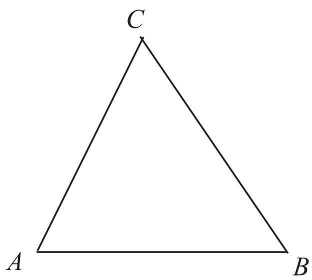
Step 4: Taking XP as the radius and X as the centre, draw a circle.



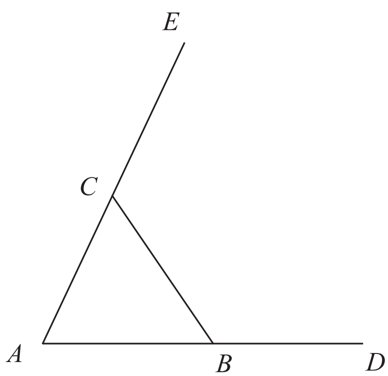
You can see that the circle which was constructed touches the three sides *AB, BC* and *CA* internally. Therefore it is called the **incircle** of the triangle *ABC*. The centre of this circle is known as the **incentre** of the triangle.

Construction of the excircle (escribed circle) of a triangle

Let us consider the triangle *ABC*.

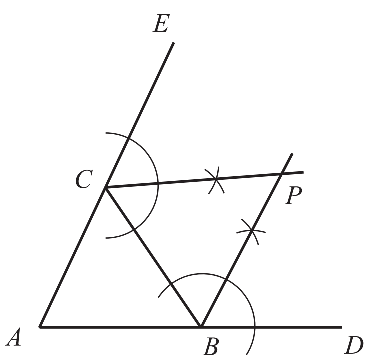


Step 1: Produce the side *AB* to *D* and the side *AC* to *E*.

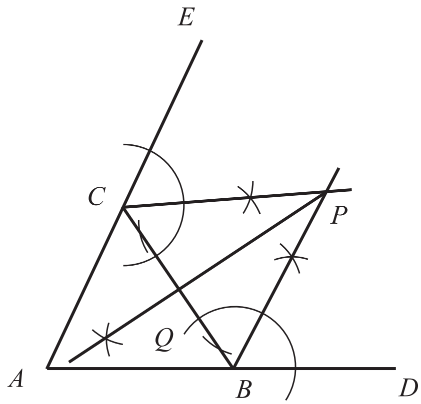


Step 2: Using the pair of compasses, construct the angle bisectors of the angles $C\hat{B}D$ and $B\hat{C}E$.

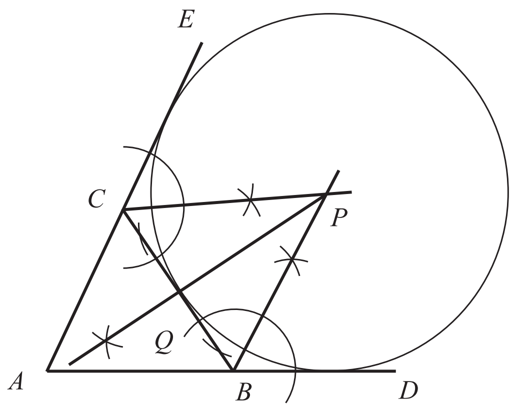
Step 3: Name the point of intersection of the two bisectors as *P*.



Step 4: Construct a perpendicular from *P* to *BC* (or to *CE* or *BD*). Name the foot of the perpendicular as *Q*.



Step 5: Taking P as the centre, draw the circle with radius PQ.



Observe that the circle touches the side BC and the sides AC and AB produced externally. Therefore it is named the **excircle** of the triangle ABC. Its centre is known as the **excentre** of the triangle.

Note: The excircle that touches the side AB and the sides CB and CA produced, as well as the excircle that touches the side CA and the sides BA and BC produced can also be constructed similarly. It is clear therefore that each triangle has three excircles.

Exercise 23.2

1. (i) Construct the triangle ABC where $AB = 5$ cm, $BC = 4.5$ cm and $AC = 4$ cm.
(ii) Construct the perpendicular bisectors of the sides BC and AC. Name the point where they meet as O.
(iii) Construct the circumcircle of the triangle ABC.

2. (i) Construct the triangle PQR where $PQ = 6$ cm, $P\hat{Q}R = 90^{o}$ and $QR = 4$ cm.
(ii) Construct the circumcircle of PQR.

3. (i) Construct the triangle XYZ where $XY = 4.2$cm $Y\hat{X}Z = 120^{o}$ and $X\hat{Y}Z = 30^{o}$.
(ii) Construct the circumcircle of XYZ.
(iii) Measure the radius of the circumcircle and write it down.

4. (i) Construct the triangle ABC where $AB = 7$ cm, $BC = 6$ cm and $AC = 5.5$ cm.
(ii) Construct the angle bisectors of the angles $A\hat{B}C$ and $B\hat{A}C$.
(iii) Name the point of intersection of the two angle bisectors as P.
(iv) Draw the incircle of the triangle ABC.

5. (i) Construct the triangle KLM where $KL = 6$ cm, $L\hat{K}M = 105^{o}$ and $KM = 9$ cm.
(ii) Construct the incircle of the triangle KLM and measure and write down its radius.

6. (i) Construct the triangle CDE where $CD = 5.5$ cm, $C\hat{D}E = 60^{o}$ and $DE = 4$ cm.
(ii) Produce CD to P where $DP = 2.8$ cm and CE to Q where $EQ = 2.5$ cm.
(iii) Construct the bisectors of $E\hat{D}P$ and $D\hat{E}Q$. Name the point where they intersect as X.
(iv) Construct a perpendicular from X to DE and name the point where it meets DE as K.
(v) Taking X as the centre, draw a circle with radius XK.

7. (i) Construct the parallelogram $ABCD$ where $AB = 6.2$ cm, $A\hat{B}C = 120^{o}$ and $BC = 4.5$ cm.
(ii) Produce the sides AB and AC and draw an excircle of the triangle ABC.
(iii) Measure and write down the radius of this circle.

23.3 Construction of a tangent to a circle

Let us recall two theorems we learnt in the lesson on tangents.

1. The straight line drawn through a point on a circle and perpendicular to the radius through the point of contact is a tangent to the circle.

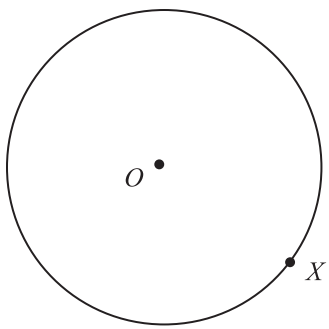
2. The tangents drawn to a circle from an external point (exterior) are equal in length.

Now let us consider how to construct tangents to circles using the above theorems.

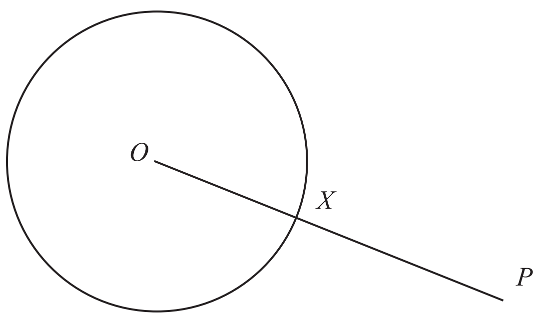
Construction of a tangent to a circle through a point on the circle

To construct this, let us use the theorem, "The straight line drawn through a point on a circle and perpendicular to the radius through the point of contact is a tangent to the circle."

Let the centre of the given circle be O and X be a point on it.

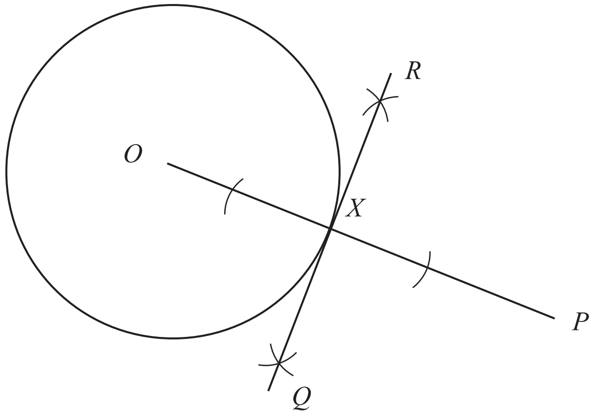


Step 1: Draw the line OX and mark a point P on OX produced.



Step 2: Using the pair of compasses, construct a perpendicular to OP through X.
To do this, use the knowledge on constructing a perpendicular to a straight line segment through a point on it.

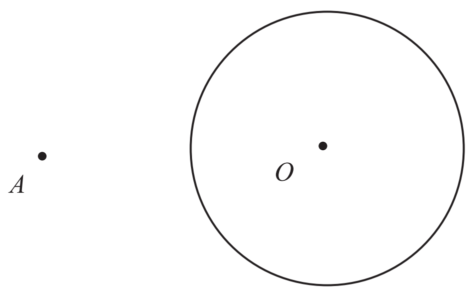
Step 3: Name that perpendicular as RQ.



RQ is the tangent drawn to the circle through X.

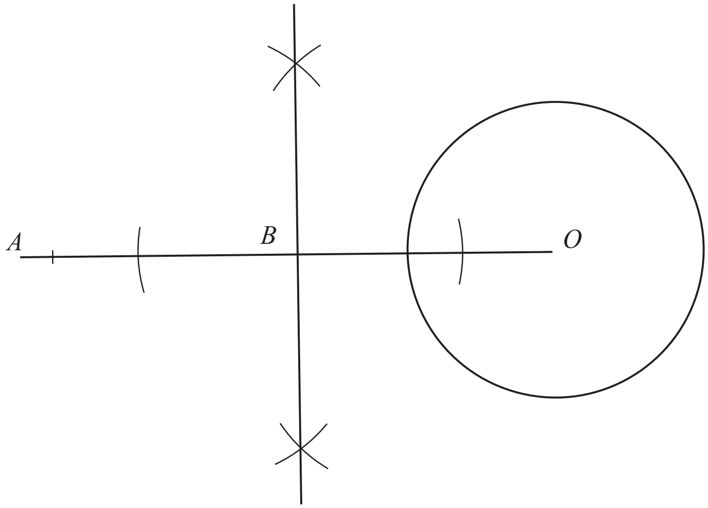
Construction of a tangent to a circle from an external point

Let O be the centre of the circle and A a point external to the circle.



To construct this, let us use the theorem, "The tangents drawn to a circle from an external point (exterior) are equal in length."

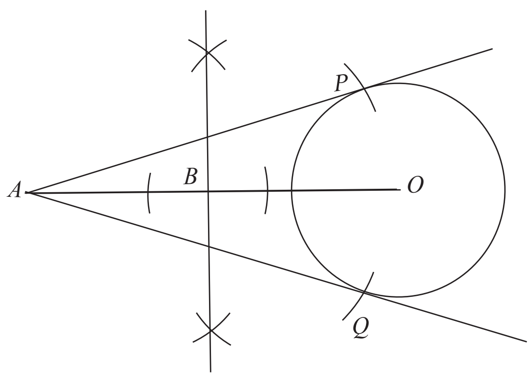
Step 1: Join OA. Construct the perpendicular bisector of the straight line segment OA. Name the point of intersection of OA and its perpendicular bisector as B. To do this construction, use the knowledge on constructing the perpendicular bisector of a straight line segment.



Step 2: Taking *B* as the centre and *BO* (or *BA*) as the radius, draw two arcs which intersect the circle.

Step 3: Name the two points of intersection as *P* and *Q*.

Step 4: Draw the lines *AP* and *AQ*.



AP and *AQ* are the tangents drawn to the circle with centre *O* from the point *A*. Using the protractor we can establish this by showing that the angles $A\hat{P}O$ and $A\hat{Q}O$ are 90º.

Exercise 23.3

1. Construct a circle of radius 3 cm. Mark a point on the circle and name it *A*. Construct a tangent to the circle through the point *A*.

2. (i) Construct a circle of radius 3.5cm and name its centre *O*. Mark a point on the circle and name it *P*. Construct a tangent to the circle through the point *P*.
(ii) Mark a point *Q* on the tangent such that $PQ = 5$cm.
(iii) Measure and write down the length of *OQ*.
(iv) Find the length of *OQ* using Pythagoras's theorem and check if your answer is correct.

3. (i) Construct the equilateral triangle *ABC* of side length 5 cm.
(ii) Construct the circle which touches *AB* at *B* and passes through the point *C*.
(iii) Measure the radius of this circle and write it down.

4. (i) Construct the circle with centre *O* and radius 2.8 cm.
(ii) Mark a point *A* on the circle and join *OA*. Mark the point *B* on *OA* produced such that $OB = 5$ cm.
(iii) Construct tangents to the circle from the point *B*.
(iv) Measure and write down the lengths of the tangents.

5. (i) Construct the triangle *ABC* such that $AB = 5$ cm, $AC = 3$ cm and $B\hat{A}C = 90^{\circ}$.
(ii) Construct the circumcircle of *ABC*.
(iii) Construct a tangent to the circle through the point *A*.
(iv) Name the point of intersection of the tangent drawn through *A* and *BC* produced as *P*.
(v) Construct another tangent to the circle from the point *P*.

6. (i) Construct a triangle *KLM* such that $KL = 9$ cm, $K\hat{L}M = 90^{\circ}$ and $LM = 4$ cm.
(ii) Construct the angle bisector of $K\hat{M}L$. Name the point where it meets *KL* as *O*.
(iii) Construct a circle taking *O* as the centre and *OL* as the radius.
(iv) Mark a point *T* on *KM* such that $ML = MT$.
(v) Find the magnitude of $O\hat{T}M$.
(vi) Draw another tangent to the circle from the point *K*.

Miscellaneous Exercise

1. (i) Construct the triangle ABC such that $AB = 6$cm, $A\hat{B}C = 45^{o}$ and $BC = 4$ cm.

(ii) Construct a line through A parallel to BC.

(iii) Construct the circle which has its centre on this parallel line and which passes through the points A and B.

2. (i) Construct the triangle PQR where $PQ = 7$cm, $P\hat{Q}R = 120^{o}$ and $QR = 4.5$ cm.

(ii) Locate the point S such that $PQRS$ is a parallelogram.

(iii) Draw the diagonal QS.

(iv) Construct the circumcircle of triangle PQS.

(v) Construct the incircle of triangle QRS.

3. Construct the triangle PQR such that $PQ = 4.8$ cm, $P\hat{Q}R = 90^{o}$ and $QR = 6.5$ cm. Construct a circle which touches PQ at P and also touches QR.

24 Sets

By studying this lesson you will be able to

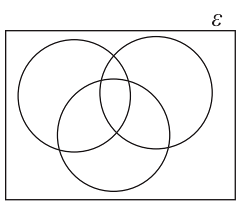
- identify the regions in a Venn diagram,
- express the regions in a Venn diagram using set notation,
- solve problems using Venn diagrams involving three sets.

Venn diagrams

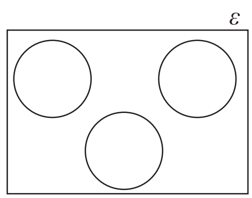
You have learnt in Grade 10 to identify regions in Venn diagrams which involve two sets and to express shaded regions in Venn diagrams using set notation. You can represent three subsets of the universal set in a Venn diagram too. Let us now consider how this is done.

The following are different ways in which three nonempty subsets of a universal set can be represented in a Venn diagram. The first figure illustrates the most general form.

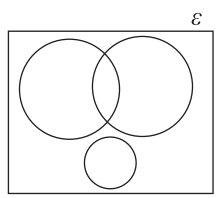
(i)



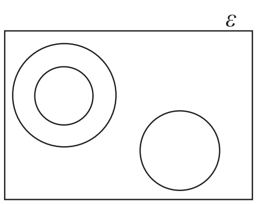
(ii)



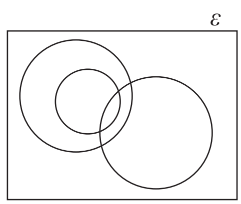
(iii)



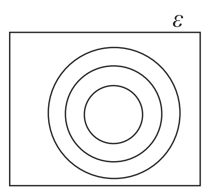
(iv)



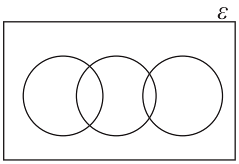
(v)



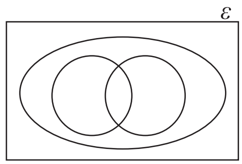
(vi)



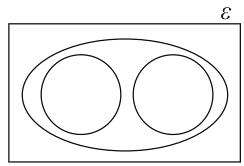
(vii)



(viii)

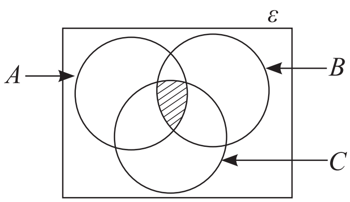


(ix)

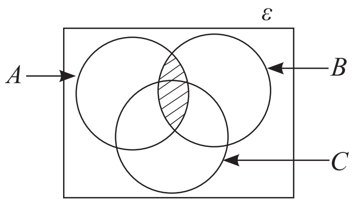


24.1 Expressing a subset denoted by a shaded region in a Venn diagram using set notation

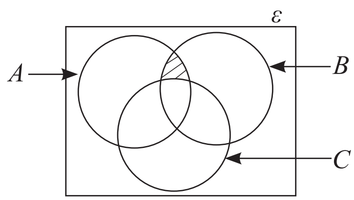
Let A, B and C be three nonempty subsets of a universal set. Several cases of shaded regions in a Venn diagram which have been expressed using set notation are given below.



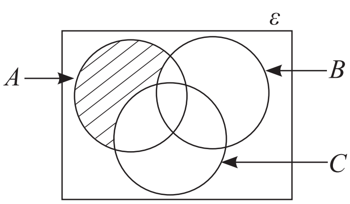
$A \cap B \cap C$



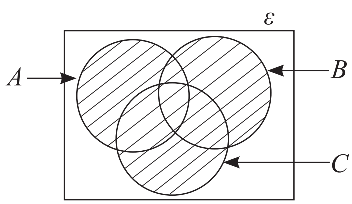
$A \cap B$



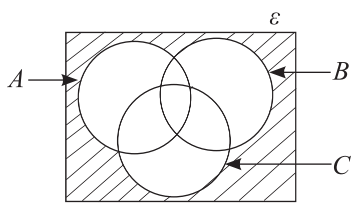
$(A \cap B) \cap C'$



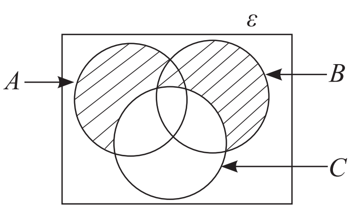
$A \cap (B \cup C)'$



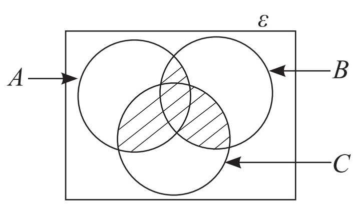
$A \cup B \cup C$



$(A \cup B \cup C)'$



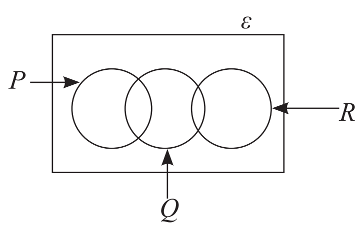
$(A \cup B) \cap C'$



$(A \cap B) \cup (B \cap C) \cup (C \cap A)$

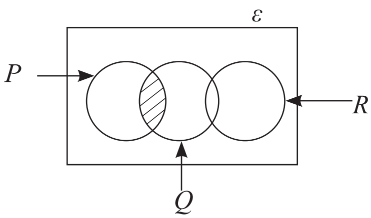
Example 1

For each of the following cases, shade the region representing the given set in a copy of the Venn diagram provided here.

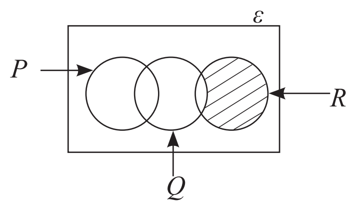


(i) $P \cap Q$ (ii) $(P \cup Q)' \cap R$ (iii) $(P \cup R)' \cap Q$
(iv) $(P \cup Q \cup R)'$

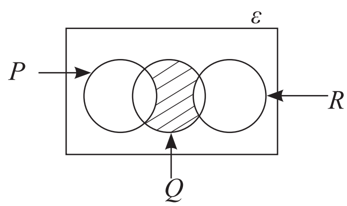
(i) $P \cap Q$



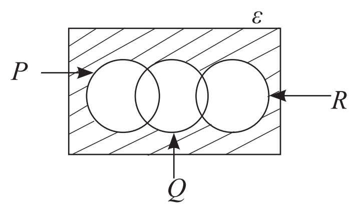
(ii) $(P \cup Q)' \cap R$



(iii) $(P \cup R)' \cap Q$

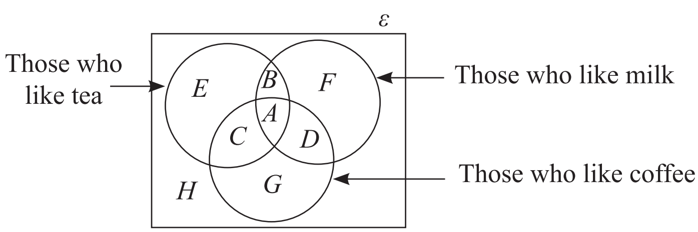


(iv) $(P \cup Q \cup R)'$



Next we will consider how to describe the nature of the elements that belong to a particular region in a Venn diagram in words. It is easier to understand this if we consider an example.

The following Venn diagram provides information on what a group of students like to drink.



The regions denoted by the capital letters of the English alphabet in the above Venn diagram can be described in words as follows.

A - Those who like to drink tea, milk and coffee

B - Those who like to drink only tea and milk. That is, those who like to drink tea and milk but do not like to drink coffee

C - Those who like to drink only tea and coffee

D - Those who like to drink only milk and coffee

E - Those who like to drink only tea

F - Those who like to drink only milk

G - Those who like to drink only coffee

H - Those who do not like to drink any of these three

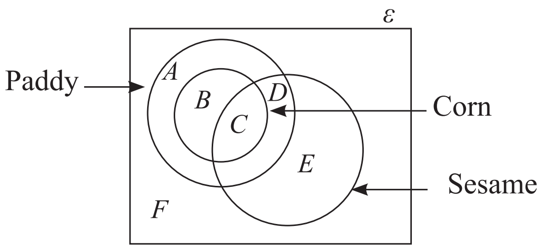
Moreover, regions that are obtained by combining two or more of the above regions can also be described in words, most often in a simple way.

A and *B* – Those who like to drink tea and milk
B and *C* and *D* – Those who like to dring exactly two of these three types
A and *B* and *C* and *D* – Those who like to drink at least two of these three types
A and *B* and *C* and *E* – Those who like to drink tea
E and *F* and *G* – Those who like to drink exactly one of these three types

Example 2

The following Venn diagram provides information on the types of crops that a group of farmers cultivate. Describe each of the subsets which are denoted by the capital letters as well as the subsets denoted by the following composite regions.

(i) *B* and *C*
(ii) *C* and *D*
(iii) *A* and *D* and *E*

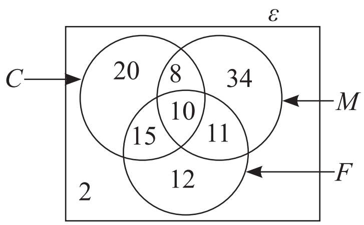


A – Farmers who cultivate only paddy
B – Farmers who cultivate only paddy and corn
C – Farmers who cultivate paddy, corn and sesame
D – Farmers who cultivate paddy and sesame but not corn
E – Farmers who cultivate only sesame
F – Farmers who do not cultivate any of these three crops
B and *C* – Farmers who cultivate corn
C and *D* – Farmers who cultivate paddy and sesame
A and *D* and *E* –Farmers who cultivate at least one crop but do not cultivate corn

Example 3

Let ε = {Families living in a Housing Scheme}
C = {Families that own a car}
M = {Families that own a motorcycle}
F = {Families that own a bicycle}

These sets have been represented in the following Venn diagram. The numbers represent the number of elements in each subset.
Answer the following questions by considering the Venn diagram.



(i) How many families own a car?
(ii) How many families own a motorcycle but neither a car nor a bicycle?
(iii) How many families do not own a bicycle?
(iv) How many families own exactly two types of vehicles?
(v) How many families own at least two types of vehicles?
(vi) How many families own exactly one of these three types of vehicles?

(i) The set of households which have cars is denoted by C. All these households need to be considered. Therefore, the number of households which have cars is $20 + 8 + 10 + 15 = 53$.

(ii) The set of households that have motor cycles is denoted by M. The households which have only motorcycles are those which have motorcycles but not cars or bicycles. Therefore, to obtain the answer, from the households which have motorcycles, those which also have either cars or bicycles have to be removed. Hence, the number of households which have only motorcycles is 34.

(iii) The number of households which do not have bicycles is obtained by removing from all the households in the scheme, those which have bicycles. In other words, these households are those which have only cars, only motor cycles, only cars and motorcycles or those which do not have any of these three types of vehicles. This number is, $20 + 8 + 34 + 2 = 64$

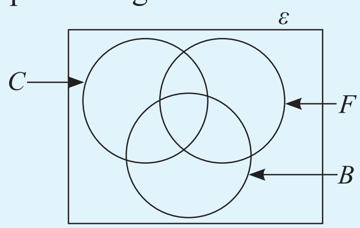
(iv) The households which have only two types of vehicles are those which have only cars and motorcycles or only cars and bicycles or only motorcycles and bicycles. This number is $15 + 8 + 11 = 34$.

(v) The households which have at least two types of vehicles are those which have either two types of vehicles or all three types of vehicles. This number is $15 + 8 + 11 + 10 = 44$.

(vi) The households which have only one type of vehicle are those which have only cars or only motorcycles or only bicycles. This number is $20 + 34 + 12 = 66$.

Exercise 24.1

1. A Venn diagram prepared based on the types of sports that each student in a group likes is given below.

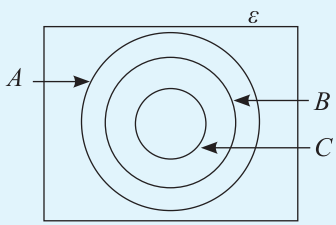


C = {Those who like cricket}
F = {Those who like football}
B = {Those who like volleyball}

By using the above Venn diagram model, shade the region that denotes each of the following sets expressed in set notation and describe it in words. Use a different Venn diagram for each of the parts (i), (ii), (iii) and (iv).

(i) $B \cap C \cap F$ (ii) $(C \cap F) \cap B'$ (iii) $(B \cup C)' \cup F$ (iv) $(B \cup C \cup F)'$

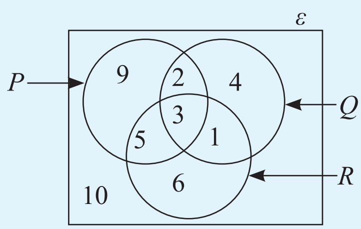
2.



By using the given Venn diagram model, shade the region that denotes each of the following sets expressed in set notation and describe it in words. Use a different Venn diagram for each of the parts (i), (ii), (iii) and (iv).

(i) $A \cap B \cap C$ (ii) $B \cap C'$
(iii) $A \cap (B \cup C)'$ (iv) $(A \cup B \cup C)'$

3.



Determine the following based on the given Venn diagram.

(i) $n\,(P \cap Q \cap R)$ (ii) $n\,(Q \cup R)'$
(iii) $n\,[(P \cap Q) \cap R')]$ (iv) $n\,[(Q \cup R)' \cap P]$
(v) $n\,(P \cup Q \cup R)'$

The numbers given in the Venn dragram are the number of elements in the corresponding region.

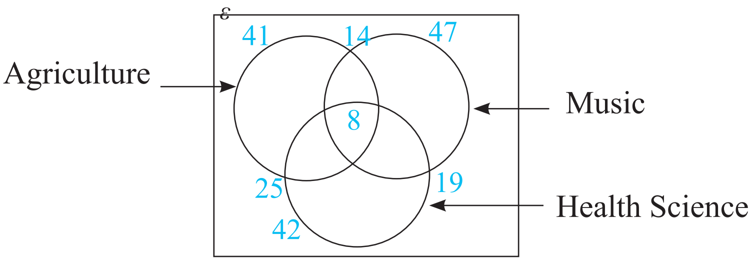
24.2 More on problems related to sets

Let us see how to solve problems related to sets by considering a few examples.

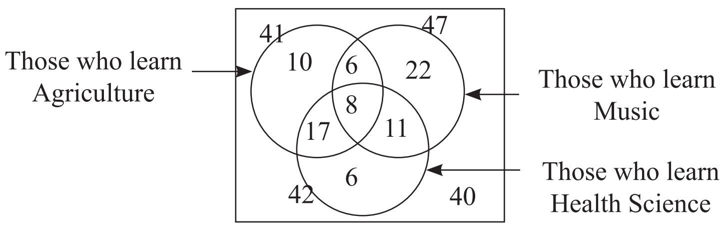
Example 1

From a group of 120 students, 41 learn agriculture, 47 learn music and 42 learn health science. 14 students learn agriculture and music, 19 learn music and health science, 25 learn agriculture and health science and 8 learn all three subjects. Represent this information in a Venn diagram and thereby determine the following.

(i) The number of students who learn only agriculture
(ii) The number of students who learn only one of these subjects
(iii) The number of students who learn at least two of these subjects
(iv) The number of students who do not learn any one of these three subjects



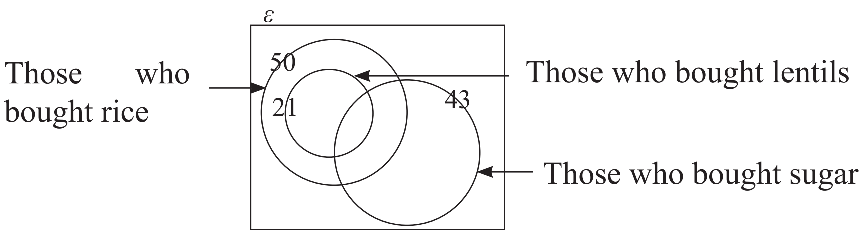
Let us find the number of elements in the remaining regions using the given information.



(i) 10
(ii) 10 + 22 + **6** = **38**
(iii) 17 + **6** + 11 + 8 = 42
(iv) 40

Example 2

According to the information gathered on the customers who visited a particular grocery store during a certain hour on a certain day, 50 bought rice, 21 bought lentils and 43 bought sugar. Moreover, everyone who bought lentils also bought rice. The given Venn diagram provides this and additional information.



(i) Twenty three customers bought rice and sugar. How many bought only sugar?

(ii) Twelve customers bought all three items. How many bought only rice and lentils?

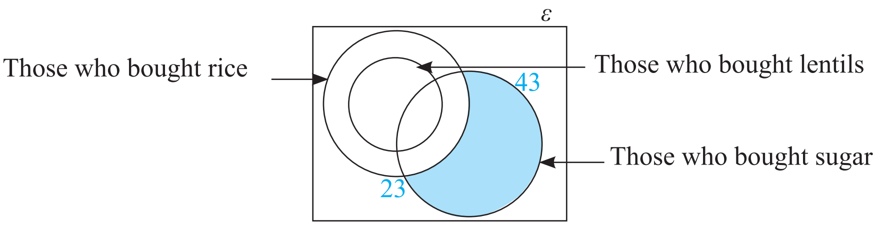
(iii) How many bought only rice?

(iv) If 90 people came to the grocery store during that hour, how many came to buy other items?

Answers

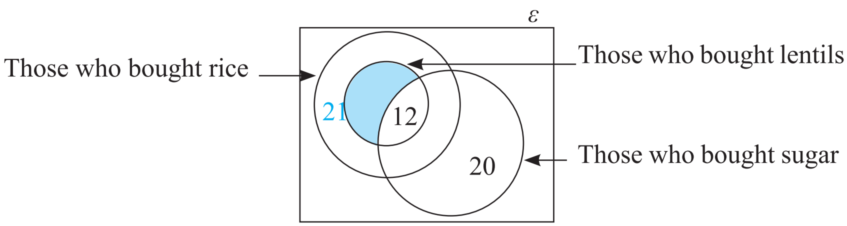
Let us find the number of elements belonging to each region using the given information.

(i)



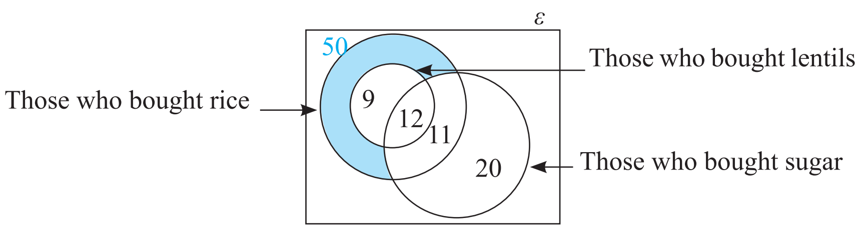
The number of customers who bought only sugar is = 43 – 23 = 20.

(ii)



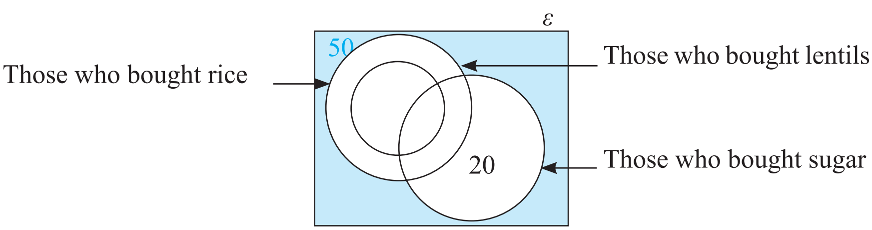
The number of customers who bought only rice and lentils is 21 – 12 = 9.

(iii)



The number of customers who bought only rice is 50 – 9 – 12 – 11 = 18.

(iv)



The number of customers who bought items other than rice, lentils and sugar = 90 – 70 = 20.

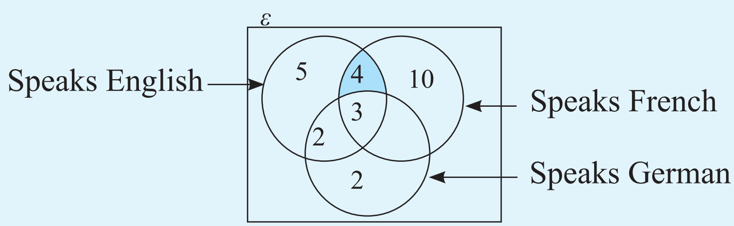
Exercise 24.2

1. Information on 20 customers who bought items from a certain stationary shop is as follows. There were 8 who bought pencils, 11 who bought pens and 13 who bought books. Of the 6 who bought pencils and books, 4 did not buy pens. There were 3 who bought pencils and pens and 3 who bought only pens. Represent this information in a Venn diagram and determine the following.
   (i) How many did not buy any of these three types of items?
   (ii) How many did not buy pens?
   (iii) What percentage of all the customers bought at least 2 of these 3 types?

2. The following information on the newspapers *A, B* and *C* that are purchased by a village community was obtained through a survey. 50% buy *A*, 67% buy *B* and 55% buy *C*. 10% buy only *A* and *B*, 15% buy only *A* and 5% buy *A* and *C* but not *B*. 17% do not buy *A* but buy *B* and *C*. Represent this information in a Venn diagram and determine the following.

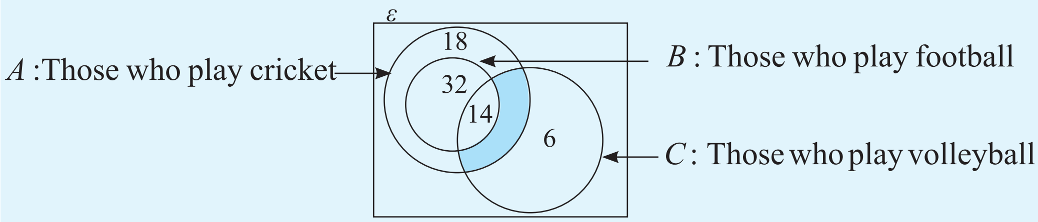
   (i) The percentage that buy all three newspapers.
   (ii) The percentage that buy *C* but not *A*.
   (iii) The percentage that buy only two of these three types.

3. The following Venn diagram has been drawn with the information that was noted down on the languages that a group of tourists who visited Sigiriya could speak.



   (i) How many can speak English?
   (ii) If 12 people in total can speak German, how many can speak French and German only?
   (iii) Describe in words the set represented by the shaded region.
   (iv) All those who could speak English stayed on with the guide who provided commentaries in English while the rest were sent with a guide who was fluent in both French and German. How many went with this guide?

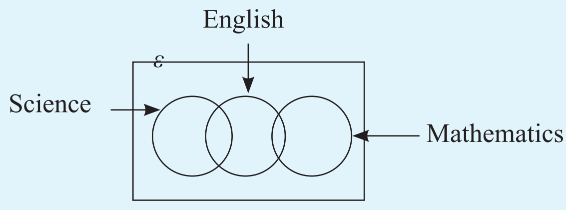
4. All the students who receive training at a certain sports academy participate in at least one of the three sports cricket, football and volley ball. The Venn diagram provides information on these students.



(i) How many students participate in all three sports?
(ii) How many students play only cricket?
(iii) Describe the group of students who are represented by the shaded region and express this set using set notation.
(iv) If 25 play volleyball, how many students belong to the shaded region?

5. Each year, 400 students are enrolled in a National College of Education. The three subjects, namely mathematics, science and physical education which are taught there are taught in both, Sinhala and English mediums.

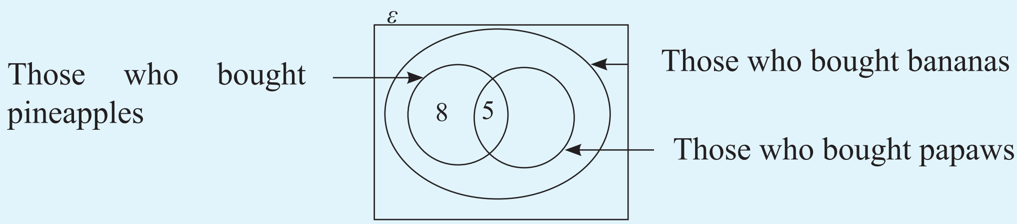
(a) Complete the following Venn diagram by marking the given information in the correct region.



(i) There are 140 students who study the subject science. 100 of these students study in the Sinhala medium.
(ii) 40 students study mathematics in the English medium.
(iii) 110 students study in the English medium
(iv) The total number of students who study mathematics is 175.

(b) (i) How many students study physical education in the English medium?
(ii) How many students study science in the English medium?
(iii) How many students study mathematics in the Sinhala medium?
(iv) Find the probability of a student selected at random from those who enrolled that year being a student who studies physical education in the Sinhala medium.

6. Information on the types of fruits that were purchased by the customers who arrived at a grocery store one day is given in the following Venn diagram. On that day, all who bought either pineapples or papaws also purchased bananas.



(i) How many bought pineapples?

(ii) If 12 people bought papaws, how many bought only papaws?

(iii) If 40 people bought bananas, how many bought only bananas?

(iv) If there were 10 other customers who purchased fruits but did not buy any one of the given three types, how many came to the grocery store that day to buy fruits?

(v) How many bought only two of these three types of fruits?

(vi) If a person was selected at random from those who came to buy fruits, what is the probability that the person purchased all three of the given types of fruits?

25 Probability

By studying this lesson you will be able to,

- solve problems involving the events of a random experiment occurring in two stages using
  (i) a grid (Cartesian plane)
  (ii) a tree diagram

Do the following exercise to recall what had been learnt in Grade 10.

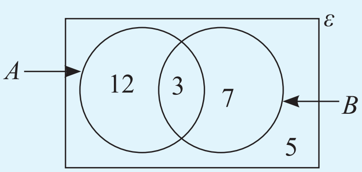
Review Exercise

1. A is an event in the sample space S of a random experiment with equally likely outcomes. If $n(A) = 23$ and $n(S) = 50$, find
  (i) $P(A)$
  (ii) $P(A')$.

2. The sample space S of a random experiment is $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Assuming that the outcomes of this experiment are equally likely, answer the questions given below.

  (i) A is a simple event of the above experiment. Write down all the events that A could be.
  (ii) For each of the above events, find $P(A)$.
  (iii) B is a compound event of the above experiment consisting of 4 elements. Write one event that B could be.
  (iv) Find $P(B)$ and $P(B')$.
  (v) X is another event with $P(X) = 0.5$ Write two events that X could be.

3. The following Venn diagram provides information on the number of elements in the subsets relevant to two events of a random experiment.
  (a) Determine the following



(i) $n(\varepsilon)$ (ii) $P(A)$ (iii) $P(B)$
(iv) $P(A \cap B)$ (v) $P(A \cup B)$ (vi) $P(A \cap B')$
(vii) $P(A' \cap B)$ (viii) $P(A \cup B)'$

4. Three identical cards numbered from 1 to 3 are in a bag. One card is drawn out at random and the number on it is checked to see if it is odd or even, and then the card is replaced in the bag. A card is drawn out again at random from the bag and the number is similarly checked to see whether it is odd or even.

(i) If the sample space is denoted by S, write S as a set and write down $n(S)$.
(ii) If A is the event of both numbers drawn being even, write A as a set and write down $n(A)$.
(iii) Hence find $P(A)$.
(iv) Represent S on a grid.
(v) If B is the event of drawing exactly one even number, then square the points on the grid that belong to B and find $P(B)$.
(vi) Represent S in a tree diagram and find the probability of drawing at least one even number.

25.1 Independent and Dependent Events

(i) Independent events

We learnt in Grade 10 that if the occurrence of an event does not depend on the occurrence of another event, then the two events are independent. If A and B are independent we know that $P(A \cap B) = P(A)P(B)$. An example of this is given below.

Let us consider the random experiment of flipping two coins simultaneously and noting the sides that are face up. It is clear that the side which is face up on one coin has no influence on the side which is face up on the other coin. Therefore the side which is face up on one coin is independent of the side which is face up on the other coin.

(ii) Dependent events

If the occurrence of an event depends on the occurrence of another event, then they are called dependent events. That is, the occurrence of one event changes the probability of the other event.

Deepen your understanding on dependent events further by studying the following examples.

a. The probability of a team winning a cricket match depends on whether the best bowler of the team plays in the match or not. Therefore the two events of the best bowler playing and the team winning are dependent events.

b. An animal is selected at random from a cattle pound in which there are cows and bulls. If the selected animal is a cow, milk can be obtained from it and if it is a bull, then milk definitely cannot be obtained from it. Therefore the event of selecting a cow and the event of obtaining milk are dependent events.

c. A bag contains 7 white balls and 3 black balls which are identical in size and shape. Let us consider the experiment of randomly drawing out a ball and noting its colour, and then, without replacing the first ball, randomly drawing out another ball, and noting its colour. As the first ball is not replaced before the second ball is drawn, the number of balls in the bag when the second one is drawn out is not 10 but 9, and the number of balls of a particular colour remaining in the bag after the first ball is drawn depends on the colour of the first ball that is drawn.

The probability of the second ball being a white ball if the first ball was white $= \frac{6}{9} = \frac{2}{3}$

The probability of the second ball being a white ball if the first ball was not white $= \frac{7}{9}$

As these two probabilities are not equal, the probability of the second ball being white is dependent on the probability of the first ball being white.

25.2 Solving problems using a grid

When a random experiment occurs in two stages, the events of the two stages may be independent or dependent. In Grade 10 we learnt how to solve problems when the events of the two stages are independent. To review what was learnt, let us consider the following example.

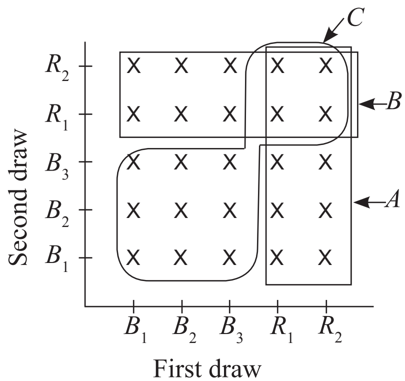
Example 1

In a bag there are 3 blue balls and 2 red balls of the same shape and size. A ball is randomly drawn out and the colour is noted. After replacing it, a ball is randomly drawn out again and its colour is also noted.

(i) Represent the sample space of this random experiment on a grid.
(ii) Find the probability of each of the following events using the grid.

(a) The first ball being red
(b) The second ball being red
(c) Both balls being red
(d) Both balls being the same colour
(e) At least one ball being red

(i) We have learnt earlier that a grid can be used to solve problems on probability only if all the outcomes of the random experiment are equally likely. Since all the balls are equal in size and shape, the probability of a ball being drawn out is the same for all the balls. Therefore the sample space can be represented on a grid and the required probabilities can be found. Let us name the three blue balls B_1, B_2 and B_3 and the two red balls R_1 and R_2.



The sample space consists of all the points that are marked by taking the horizontal axis to represent the outcomes of the first draw and the vertical axis to represent the outcomes of the second draw.

Since the second ball is taken out after replacing the first ball, the two events are independent of each other.

The probability of an event is found using a grid, by dividing the number of points relevant to the given event by the total number of points in the sample space.

(ii) The event of drawing a red ball on the first occasion is denoted by A on the grid. 10 points from the sample space belong to A. The whole sample space consists of 25 points.

$\therefore$ the probability of the first ball being red $= \dfrac{\text{number of points in } A}{\text{number of points in the sample space}}$

$$= \frac{10}{25} = \frac{2}{5}$$

(b) The event of the second ball being red is denoted by B on the grid.

Accordingly,

the probability of the second ball being red $= \dfrac{\text{number of points in } B}{\text{number of points in the sample space}}$

$$= \frac{10}{25} = \frac{2}{5}$$

(c) The event of both balls being red is the set of points which are common to both A and B. There are 4 points in this set.

$$\therefore \text{ the probability of both balls being red} = \frac{\text{number of points common to both } A \text{ and } B}{\text{number of points in the sample space}}$$

$$= \frac{4}{25}$$

(d) For both balls to be the same colour, they should both be blue or both be red. The set of points relevant to this event is denoted by C. There are 13 points in this set.

$$\therefore \left.\begin{array}{l}\text{the probability of both balls being} \\ \text{the same colour}\end{array}\right\} = \frac{\text{number of points in } C}{\text{number of points in the sample space}}$$

$$= \frac{13}{25}$$

(e) At least one ball being red means that one ball is red or both balls are red. This means all the points in both A and B. There are 16 points in total.

$$\therefore \text{ the probability of at least one ball being red} = \frac{16}{25}$$

Now let us consider an example of a random experiment consisting of two stages, where the events are dependent.

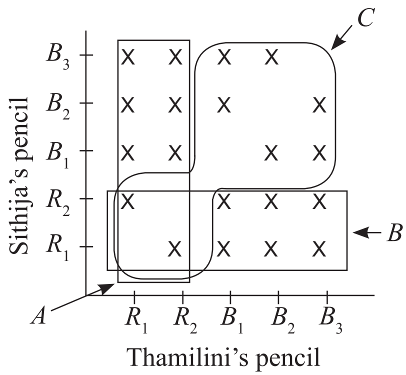
Example 2

In Sithija's pencil box there are 2 red pencils and 3 blue pencils of the same size and shape. Sithija randomly draws one pencil out and gives it to his friend Thamilini. Then Sithija randomly draws another pencil out for himself.

(i) Write the sample space in terms of the outcomes and then represent it on a grid.
(ii) Find the probability of each of the following events using the grid.
- (a) Drawing a red pencil for Thamilini
- (b) Sithija drawing a red pencil for himself
- (c) Both getting pencils of the same colour
- (d) Only Thamilini getting a red pencil

(i) Let us name the three blue pencils B_1, B_2, B_3 and the two red pencils R_1, R_2.The pencil given to Thamilini is one of R_1, R_2, B_1, B_2, B_3 and the pencil Sithija drew for himself is also one of these. However since Sithija cannot have the pencil which was given to Thamilini, the events (R_1, R_1), (R_2, R_2), (B_1, B_1), (B_2, B_2) and (B_3, B_3) cannot occur. Apart from these 5 points, the remaining 20 belong to the sample space. Therefore the sample space is the set

$\{(R_1, R_2), (R_1, B_1), (R_1, B_2), (R_1, B_3), (R_2, R_1), (R_2, B_1)...\}$. This sample space can be represented on a grid as follows.



(a) The 8 points corresponding to Thamilini receiving a red pencil is marked as A .

$\therefore$ probability of Thamilini receiving a red pencil $= \frac{8}{20} = \frac{2}{5}$

(b) The 8 points corresponding to Sithija getting a red pencil is marked as B.

$\therefore$ probability of Sithija getting a red pencil $= \frac{8}{20} = \frac{2}{5}$

(c) The set of points corresponding to both getting pencils of the same colour is marked as C. This is the event of either both getting red pencils or both getting blue pencils. C also has 8 points.

$\therefore$ probability of both getting pencils of the same colour $= \frac{8}{20} = \frac{2}{5}$

(d) For only Thamilini to get a red pencil, Sithija has to get a blue pencil while Thamilini gets a red pencil. There are 6 points corresponding to this.

$\therefore$ probability of only Thamilini getting a red pencil $= \frac{6}{20} = \frac{3}{10}$

Exercise 25.1

1. In a box, there are 2 white balls and 4 red balls of the same shape and size. One ball is drawn out randomly from this box and its colour is noted.

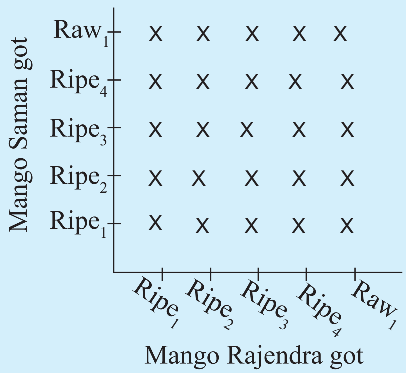
(a) Write down the sample space S of all possible equally likely outcomes.

(b) If the ball is put back into the box and a ball is randomly drawn out again and its colour is also noted, draw the sample space of the equally likely outcomes of this experiment on a grid.

(c) If the ball is not put back into the box and another ball is randomly drawn out and its colour is noted, draw the sample space of this experiment on a grid.

(d) For the experiments described in (*b*) and (*c*) above, find separately, the probability of the two balls taken out being of the same colour.

2. In a bag there are 4 ripe mangoes and 1 raw mango, all of the same size and shape. Saman randomly takes one mango from this bag and gives it to his friend Rajendra. After that Saman randomly takes out a mango for himself. A grid with the sample space prepared by Saman is given below.



(a) There is an error in the above sample space. Draw the correct sample space on another grid.

(b) Using the correct sample space, find the probability of each of the following events.

(i) Both getting ripe mangoes

(ii) Only Rajendra getting a ripe mango

(iii) Only one person getting a ripe mango

(c) Rajendra says that at least one person is sure to get a ripe mango. Explain with reasons whether this is true or not.

3. Sarath who was preparing for a journey, randomly took out two shirts (one after the other) from a box containing 4 white shirts and 3 black shirts.

(a) Represent the sample space on a grid by denoting the white shirts in the box by W_1, W_2, W_3, W_4 and the black shirts by B_1, B_2, B_3.

(b) Using the grid, find the probability of each of the following events.

(i) Both shirts being white
(ii) Only one shirt being white
(iii) At least one shirt being white

4. In a dish there were 3 milk toffees, 2 orange flavoured toffees and 1 tamarind flavoured toffee which were all the same size and shape. Sandaru randomly took one of these toffees and tasted it. Later she randomly took another one and gave it to her friend Jessie.

(a) By taking into consideration the flavours of the toffees, represent the relevant sample space of equally likely outcomes on a grid.

(b) Using the grid, find the probability of each of the following events.

(i) Both getting toffees of the same flavour.
(ii) Only one person getting a milk toffee.
(iii) Jessie getting a tamarind flavoured toffee.

25.3 Solving problems using tree diagrams

A tree diagram can be used to find the probabilities of events related to a random experiment having many stages. In this lesson we will consider only random experiments having two stages. Let us study this by considering the following examples.

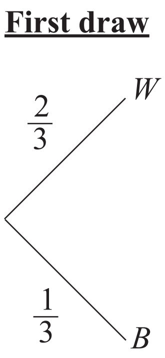
In Grade 10 you learnt to find probabilities when the events were independent. Let us review what you learnt earlier.

Example 1

In a bag there are two white balls and one black ball of the same size and shape. One ball is randomly drawn out and its colour is noted. After replacing it, a ball is randomly drawn out again and its colour is also noted.

(i) Represent the sample space of this random experiment in a tree diagram.
(ii) Find the probability of each of the following events using the tree diagram.
(a) Drawing a white ball on both occasions
(b) Drawing a white ball on the first occasion
(c) Drawing only one white ball
(d) Drawing at least one white ball

(i) Let us denote the event of drawing a white ball by W and drawing a black ball by B. As the outcomes are equally likely, the probability of drawing a white ball out the first time is $\frac{2}{3}$ and a back ball out the first time is $\frac{1}{3}$. In that part of the tree diagram which represents the first draw, let us indicate the probability of each of these two events on the relevant branches.

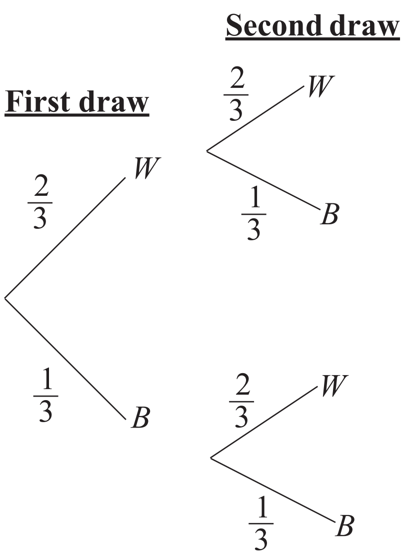


The sum of the probabilities on the two branches = $\frac{2}{3} + \frac{1}{3}$

$= 1$

Note: The sum of the probabilities on the branches which originate from a single point is 1. This occurs at every stage.

Now let us extend the above tree diagram to indicate the probabilities of the second stage.



As the second ball is drawn out after replacing the first ball, the number of balls in the bag when the second ball is drawn out is the same as the number that was in the bag initially. Therefore the probabilities of drawing a white ball and drawing a black ball on the second occasion are the same as those on the first occasion. These probabilities are marked on the appropriate branches.

Notice that the sum of the probabilities on the branches that originate from a point, all add up to

(ii) When both occasions are considered there are 4 possible outcomes.

Outcome	**Probability**	
(W, W)	$\frac{2}{3} \times \frac{2}{3}$	$\frac{4}{9}$
(W, B)	$\frac{2}{3} \times \frac{1}{3}$	$\frac{2}{9}$
(B, W)	$\frac{1}{3} \times \frac{2}{3}$	$\frac{2}{9}$
(B, B)	$\frac{1}{3} \times \frac{1}{3}$	$\frac{1}{9}$

As an example, (W, W) represents the event of both balls drawn being white. Its probability is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$. The reason why these two can be multiplied together is because they are independent events. The four events (W, W), (W, B), (B, W) and (B, B) are mutually exclusive events. The reason for this is because, no two of these four events can occur together. The required probabilities can be calculated as follows.

(a)Probability of drawing a white ball on the first occasion and a white ball on the second occasion too

$$= P(W, W)$$
$$= \frac{4}{9} \text{ (from the table)}$$

(b) Probability of drawing a white ball on the first occasion $= P(W, W) + P(W, B)$

$$= \frac{4}{9} + \frac{2}{9} = \frac{6}{9} = \frac{2}{3}$$

(c) Probability of drawing only one white ball $= P(W, B) + P(B, W)$

$$= \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$$

(d) Probability of drawing at least one white ball $= P(W, W) + P(W, B) + P(B, W)$

$$= \frac{4}{9} + \frac{2}{9} + \frac{2}{2} = \frac{8}{9}$$

Note: The answer in (d) can be obtained from $1 - P(B, B)$ too.

Now let us consider an example where the events are dependent.

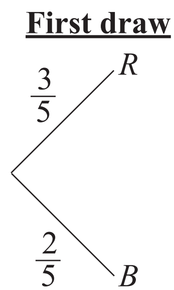
Example 2

In a bag there are 3 red balls and 2 blue balls of the same size and shape. One ball is randomly drawn out and its colour is noted. Without replacing the first ball, a second ball is randomly drawn out and its colour too is noted.

(i) Represent the sample space in a tree diagram.

(ii) Using the tree diagram find the probability of each of the following events.
  (a) Drawing a red ball on both occasions
  (b) Drawing only one red ball
  (c) Drawing at least one red ball

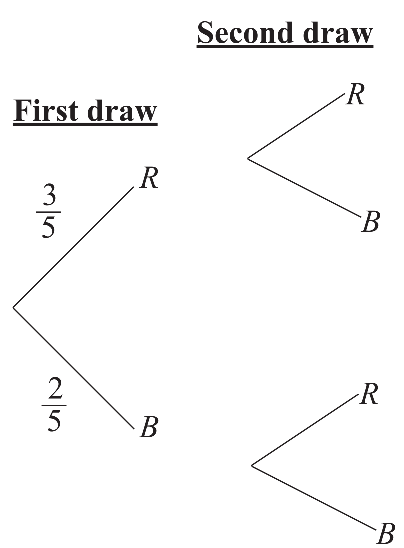
(i) The initial part of the tree diagram is shown below.



Here R represents the event of drawing a red ball while B represents the event of drawing a blue ball. As there are 3 red balls and 2 blue balls in the bag,

$P(R) = \frac{3}{5}, P(B) = \frac{2}{5}$.

Now let us extend the above tree diagram and include the second draw also.



How the probabilities relevant to the second draw were found can be described as follows.

The probabilities on the branches in the second stage are different to those on the branches in the first stage. This is because the probabilities for the second stage have to be found after considering the first stage.

If the first ball drawn is red, then the bag will have 2 red balls and 2 blue balls remaining.

$\therefore$ the probability of the second ball drawn being red $= \frac{2}{4}$

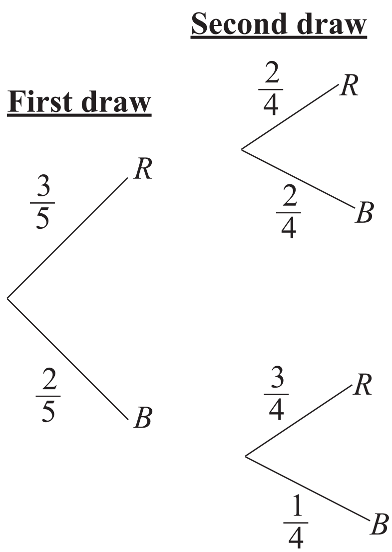
The probability of the second ball drawn being blue $= \frac{2}{4}$

If the first ball drawn is blue, the bag will have 3 red balls and 1 blue ball remaining.

$\therefore$ the probability of the second ball drawn being red $= \frac{3}{4}$

The probability of the second ball drawn being blue $= \frac{1}{4}$

Let us write these probabilities on the relevant branches of the tree diagram, and complete the given outcomes table. Establish that the sum of the probabilities along the four branches is 1.



Outcome	Probability	
(R, R)	$\frac{3}{5} \times \frac{2}{4}$	$\frac{6}{20}$
(R, B)	$\frac{3}{5} \times \frac{2}{4}$	$\frac{6}{20}$
(B, R)	$\frac{2}{5} \times \frac{3}{4}$	$\frac{6}{20}$
(B, B)	$\frac{2}{5} \times \frac{1}{4}$	$\frac{2}{20}$

In the table, the probability of the outcome (R, R) (drawing two red balls) has been found by multiplying the relevant probabilities. But these two probabilities are not independent. This is because the probability of drawing a red ball on the second occasion depends on whether or not a red ball was drawn on the first occasion. Here however, the probability of drawing a red ball on the second occasion has been found by assuming that the first ball drawn is a red ball. Therefore to find the probability of (R, R), the two relevant probabilities can be multiplied.

The events (R, R), (R, B), (B, R), (B, B) in the above table are mutually exclusive. Therefore to find the probability of a certain event using the tree diagram, we need to select the outcomes relevant to the event from the table and add the probabilities of these outcomes.

(a) Probability of drawing two red balls $= P\,(R, R)$

$$= \frac{6}{20} = \frac{3}{10}$$

(b) Probability of drawing only one red ball $= P\,(R, B) + P\,(B, R)$

$$= \frac{6}{20} + \frac{6}{20} = \frac{12}{20} = \frac{3}{5}$$

(c) Probability of drawing at least one red ball $= P\,(R, B) + P\,(B, R) + P\,(R, R)$

$$= \frac{6}{20} + \frac{6}{20} + \frac{6}{20} = \frac{18}{20} = \frac{9}{10}$$

Note: the answer in (c) can be obtained by using $1 - P\,(B, B)$ too.

Exercise 25.2

1. In a box with 10 bulbs of the same type, 3 are known to be faulty. Nimal randomly draws out one bulb from the box and checks whether it is faulty. Without replacing the first bulb he draws out another bulb and checks whether it is faulty.

(i) Represent the sample space of the above random experiment in a tree diagram.

(ii) Nimal states that the events of the first bulb being faulty and the second one being faulty are dependent events. Giving reasons explain whether this statement is true or false.

(iii) Find the probability of each of the following events using the tree diagram.

(a) Both bulbs being faulty

(b) Exactly one bulb being faulty

(c) At least one bulb being faulty

2. The probability of the football player A of a certain team playing in a match is $\frac{3}{4}$. If A plays in the match, then the probability of his team winning is $\frac{5}{8}$ and if he doesn't play, then his team winning or losing is equally likely. This match is either won or lost.

(i) Find the probability of A not playing in the match.

(ii) Find the probability of A's team winning despite him not playing in the match.

(iii) Represent the sample space in a tree diagram, by taking A playing in a match or not as the first stage and the team winning or losing as the second stage.

(iv) Find the probability of A's team winning using the tree diagram.

(v) Giving reasons state whether it is more advantageous or not for the team, if A plays in the match.

3. In a bag there are 4 ripe wood apples and 3 unripe wood apples of the same size and shape. Namali randomly draws out a fruit from the bag. If it is a ripe fruit, she again randomly draws out another one without replacing the first. If the first fruit is unripe, she puts it back into the bag and randomly draws out another fruit.

(i) Represent the sample space of this random experiment in a tree diagram.

(ii) From the following statements made by Namali, with reasons, state which ones are true.

(a) "The first fruit being ripe and the second fruit being ripe are two independent events"

(b) "The first fruit being unripe and the second fruit being unripe are two dependent events"

(iii) Find the probability of each of the following events using the tree diagram.
(a) Both fruits being ripe
(b) Second fruit being ripe
(c) From the two fruits, only one being ripe

4. In Sirimal's cattle pound, there are 5 bulls and 15 cows. In Nadan's cattle pound, there are 2 bulls and 8 cows. Sirimal and Nadan agree to exchange one animal each. After Sirimal randomly selects one animal and sends it to Nadan, Nadan randomly selects one animal and sends it to Sirimal.

(i) Draw the relevant sample space in a tree diagram.
(ii) Using it, find the probability of each of the following events.
(a) There being a reduction in the number of bulls in Sirimal's pound because of the exchange
(b) There being an increase in the number of bulls in Sirimal's pound because of the exchange
(c) There being no difference in the number of bulls and cows in each of the two pounds because of the exchange
(iii) Now suppose they exchange animals in a way different to that mentioned above. Suppose Sirimal and Nadan randomly take one animal from each of their pounds and go to their friend Abdul's house and exchange the cattle there and bring the exchanged cattle back to the pound. Find the probabilities of the events given in (ii) above for this random experiment.

5. X and Y are two drugs given for the same illness which have 90% and 80% effectiveness respectively. If a person does not recover from the illness by using one of the drugs, then the other one is also given. If this too fails, then a surgery is done.

(i) Find the probability of the patient recovering after both drugs are given.
(ii) Find the probability of a patient having to undergo surgery.

6. Information on the clerks and labourers who work in an institute is given in the following table.

Position / Gender	Male	Female	Total
Clerk	5	8	13
Labourer	2	1	3
Total	7	9	16

(i) Find the probability of a person taken randomly from this institute being,
   (a) a labourer
   (b) a clerk

(ii) A clerk and a labourer are randomly taken from this institute.
   (a) Write all the relevant probabilities on the branches of a tree diagram.
   (b) Using the tree diagram, find the probability of at least one of those chosen being male.

7. In a box there are 2 white balls and 1 black ball of the same shape and size. From this box, a ball is drawn out randomly and discarded and then, another is drawn out randomly. Find the probability of at least one of the two drawn balls being white.

8. In box *A* there are 3 blue marbles and 2 red marbles of the same shape and size. In box *B* there are 4 blue marbles and 5 red marbles of the same shape and size as those in box *A*. A marble is drawn out randomly from *A* and placed in *B*. Then a marble is drawn out randomly from *B* and placed in *A*. Find the probability that there has been no change in the colour composition of the marbles in *A*.

9. In grade 11 of a certain school there are three parallel classes. The number of children in these three classes is in the ratio 2: 2: 3. The teachers who teach mathematics to the three classes are *A, B* and *C* respectively. The principal makes the following statement based on his experience. "90% of the children in the class taught by *A*, 80% of the children in the class taught by *B* and 60% of the children in the class taught by *C* will pass the forthcoming examination."

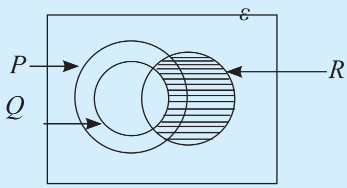
(i) Find the probability of a randomly chosen child from grade 11 passing the examination, based on the above statement.

(ii) Using the above answer, evaluate the percentage of students who will pass.

Revision Exercise – Term 3

Part I

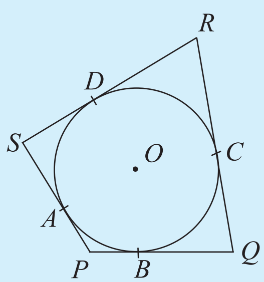
1. Solve the following inequality and represent the solutions on a number line.
$2x + 5 \leq 15$

2. 

Express the shaded region in the Venn diagram using set notation.

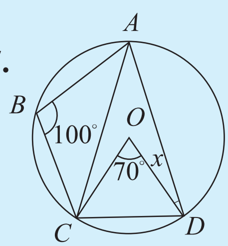
3. The area of the square drawn on the hypotenuse of a right angled isosceles triangle is 64 cm^2. Find the area of a square drawn on one of the other sides.

4. Find p and q if $\begin{pmatrix} -3 & 2 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} p \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ q \end{pmatrix}$.

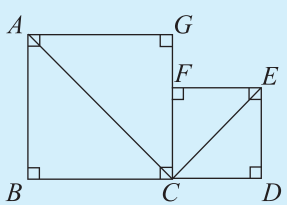
5. 

The tangents drawn to the circle with centre O, at the points A, B, C and D which lie on the circle, meet at the points P, Q, R and S as shown in the figure. If $PQ + SR = 20$ cm, find the perimeter of the quadrilateral $PQRS$.

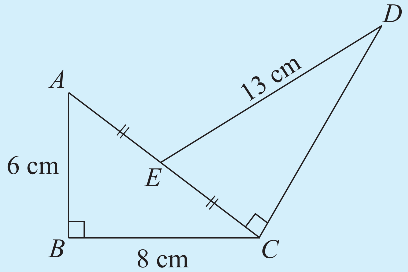
6. A and B are two events of a random experiment such that $P(A) = 0.4$ and $P(A \cup B) = 0.7$. If A and B are independent events, find the value of $P(B)$.

7. 

In the circle with centre O shown in the figure, $C\hat{O}D = 70^\circ$ $C\hat{B}A = 100^\circ$. Find the magnitude of $O\hat{D}A$.

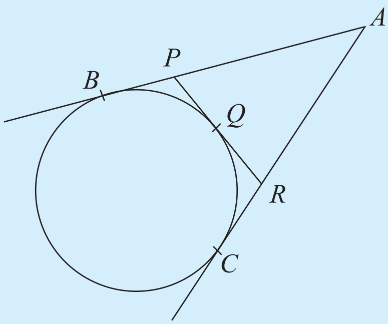
8. 

$ABCG$ and $FCDE$ in the figure are squares. If $AC^2 = 12$ cm^2 and $CE^2 = 6cm^2$, find the area of the whole figure.

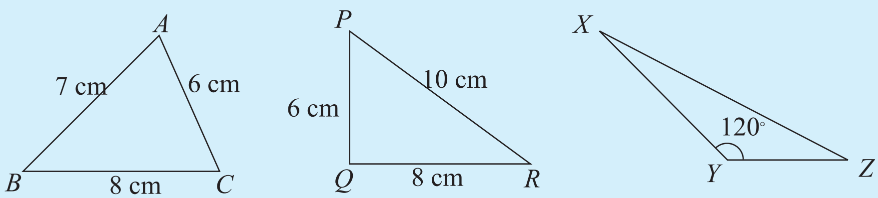
9. 

The triangles ABC and ECD in the figure are right angled triangles. Find the area of the whole figure.

10. Write down the matrix $-2A$ if $A = \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}$

11. 

The tangents drawn from the point A to the circle in the figure are AB and AC. The tangent drawn to the circle at the point Q meets AB and AC at P and R respectively. If the perimeter of the triangle APR is 18 cm, find the length of AB.

12. 

(i) (ii) (iii)

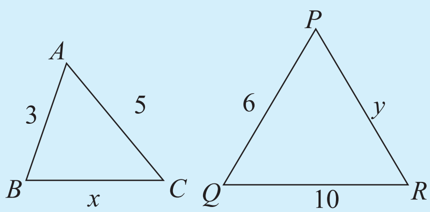
Write down the number of the figure suitable for the blank space.

(a) The circum-centre of the triangle lies on a side of the triangle, in triangles of the form ....

(b) The circum-centre of the triangle lies in the exterior of the triangle, in triangles of the form ....

(c) The circum-centre of the triangle lies in the interior of the triangle, in triangles of the form ....

13. Find the values of x and y in the given equiangular traingles.



14. Represent the integral solutions of the inequality $4x + 3 \geq 8$ on a number line.

15. Write down the coordinates of the turning point of the graph of the function $y = x^2 + 5x + 9$ without drawing the graph.

Part II

1. $A\hat{B}C = 90^\circ$ in the right angled triangle ABC.

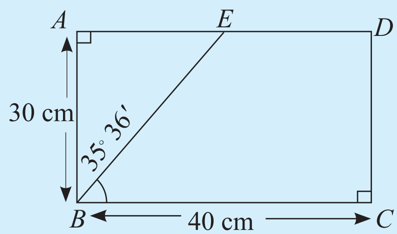
(i) If P is the midpoint of the side BC, then show that $4\,(AP^2 - AB^2) = BC^2$.

(ii) If Q is the midpoint of the side AB, then show that $4\,(CQ^2 - BC^2) = AB^2$.

(iii) Using the results obtained in (i) and (ii) above, deduce that $4\,(AP^2 + CQ^2) = 5\,AC^2$.

(iv) Show using the result obtained in (iii) above, that when the above triangle is an isosceles right angled triangle, then $AP{:}QP = \sqrt{5} : \sqrt{2}$.

2. (a)



$ABCD$ is a rectangle. Using the trigonometric tables

(i) find the length of AE.

(ii) Calculate the perimeter of the trapezium $BCDE$.

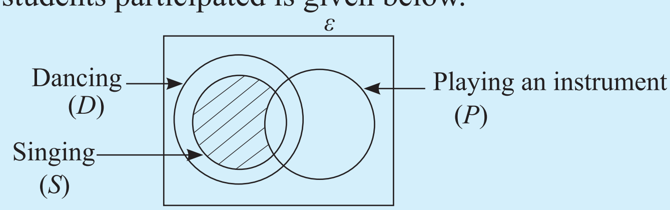
(b) The three cities A, B and C are located such that city B is 50 km from city A, on a bearing of 040° from city A, and city C is directly to the North of city A and on a bearing of 270° from city B.

(i) Draw a suitable sketch and mark the above information in it.

(ii) Find the distance from city A to city C.

(iii) It is necessary to construct a large water tank on top of a concrete pillar to provide water to these three cities. In the above sketch, mark a suitable place where the tank can be built, so that the lengths of the water pipes that carry water from the tank to each of the three cities is the same, and mark this location as T in the sketch.

3. Information on some of the students who were involved in a pageant in which 160 students participated is given below.



$\frac{1}{4}$ of all those who participated in the pageant were involved in at least one of the three activities, dancing, singing and playing an instrument. From the 16 students who were involved in playing an instrument and dancing, 6 students also sang. Twice the number of those who were involved in playing an instrument only was involved in singing and dancing only, and five times the number of those who were involved in playing an instrument only, danced.

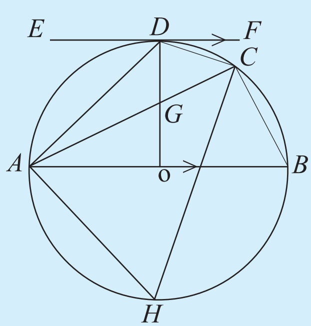
Copy the given Venn diagram in your exercise book and answer the following questions.

(i) Mark the given information accurately in the Venn diagram. How many students were involved in all three activities, singing, dancing and playing an instrument?
(ii) How many were involved in only playing an instrument?
(iii) Express the number that was involved in only one of these three activities as a fraction of the total number of students that participated.
(iv) Describe the activities in which the students in the set represented by $(S' \cap D) \cap P$ (v) were involved. How many students belong to this set?

4. Identical balls of different colours have been placed in two vessels A and B. Vessel A has 3 black balls and 2 white balls. Vessel B has 2 black balls and 3 white balls. A person randomly draws out a ball from vessel A and places it in vessel B. He then randomly draws out a ball from vessel B.

(i) Draw a tree diagram with the probabilities relevant to the above events marked on the branches.
(ii) Using the tree diagram, find the probability of drawing out balls of the same colour on both occasions.

5.



As indicated in the figure, AB is a diameter of the circle with centre O. The tangent EF drawn to the circle at the point D is parallel to AB.

(i) Write down two angles which are equal to $A\hat{B}D$.
(ii) Find the magnitude of $E\hat{D}O$.
(iii) Show that OBCG is a cyclic quadrilateral.

6. Do the following constructions using a pair of compasses and a straight edge with a mm/cm scale and showing the construction lines clearly.

(i) Construct the triangle ABC such that $AB = 8$ cm, $A\hat{B}C = 90^\circ$ and $BC = 4$ cm.

(ii) Construct the trapezium $ABCD$ such that $DC = 2$ cm and DC is parallel to AB.

(iii) Construct the circle that externally touches CB produced at G, CA produced at E and AB produced at F.

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மடக்கைகள்

LOGARITHMS

											මධ්‍යන්‍ය අන්තරය / இடை வித்தியாசங்கள் / Mean Differences								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

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மடக்கைகள்
LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

ප්‍රකෘති සයින
இயற்கைச் சைன்கள்
NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
0	0.0000	0.0029	0.0058	0.0087	0.0116	0.0145	0.0175	89	3	6	9	12	15	17	20	23	26
1	.0175	.0204	.0233	.0262	.0291	.0320	.0349	88	3	6	9	12	15	17	20	23	26
2	.0349	.0378	.0407	.0436	.0465	.0494	.0523	87	3	6	9	12	15	17	20	23	26
3	.0523	.0552	.0581	.0610	.0640	.0669	.0698	86	3	6	9	12	15	17	20	23	26
4	.0698	.0727	.0756	.0785	.0814	.0843	.0872	85	3	6	9	12	15	17	20	23	26
5	0.0872	0.0901	0.0929	0.0958	0.0987	0.1016	0.1045	84	3	6	9	12	14	17	20	23	26
6	.1045	.1074	.1103	.1132	.1161	.1190	.1219	83	3	6	9	12	14	17	20	23	26
7	.1219	.1248	.1276	.1305	.334	.1363	.1392	82	3	6	9	12	14	17	20	23	26
8	.1392	.1421	.1449	.1478	.1507	.1536	.1564	81	3	6	9	11	14	17	20	23	26
9	.1564	.1593	.1622	.1650	.1679	.1708	.1736	80°	3	6	9	11	14	17	20	23	26
10°	0.1736	0.1765	0.1794	0.1822	0.1851	0.1880	0.1908	79	3	6	9	11	14	17	20	23	26
11	.1908	.1937	.1965	.1994	.2022	.2051	.2079	78	3	6	9	11	14	17	20	23	26
12	.2079	.2108	.2136	.2164	.2193	.2221	.2250	77	3	6	9	11	14	17	20	23	26
13	.2250	.2278	.2306	.2334	.2363	.2391	.2419	76	3	6	8	11	14	17	20	23	25
14	.2419	.2447	.2476	.2504	.2532	.2560	.2588	75	3	6	8	11	14	17	20	23	25
15	0.2588	0.2616	0.2644	0.2672	0.2700	0.2728	0.2756	74	3	6	8	11	14	17	20	22	25
16	.2756	.2784	.2812	.2840	.2868	.2896	.2924	73	3	6	8	11	14	17	20	22	25
17	.2924	.2952	.2979	.3007	.3035	.3062	.3090	72	3	6	8	11	14	17	19	22	25
18	.3090	.3118	.3145	.3173	.3201	.3228	.3256	71	3	6	8	11	14	17	19	22	25
19	.3256	.3283	.3311	.3338	.3365	.3393	.3420	70	3	5	8	11	14	16	19	22	25
20°	0.3420	0.3448	0.3475	0.3502	0.3529	0.3557	0.3584	69	3	5	8	11	14	16	19	22	25
21	.3584	.3611	.3638	.3665	.3692	.3719	.3746	68	3	5	8	11	14	16	19	22	24
22	.3746	.3773	.3800	.3827	.3854	.3881	.3907	67	3	5	8	11	13	16	19	21	24
23	.3907	.3934	.3961	.3987	.4014	.4041	.4067	66	3	5	8	11	13	16	19	21	24
24	.4067	.4094	.4120	.4147	.4173	.4200	.4226	65	3	5	8	11	13	16	19	21	24
25	0.4226	0.4253	0.4279	0.4305	0.4331	0.4358	0.4384	64	3	5	8	10	13	16	18	21	24
26	.4348	.4410	.4436	.4462	.4488	.4514	.4540	63	3	5	8	10	13	16	18	21	23
27	.4540	.4566	.4592	.4617	.4643	.4669	.4695	62	3	5	8	10	13	15	18	21	23
28	.4695	.4720	.4746	.4772	.4797	.4823	.4848	61	3	5	8	10	13	15	18	20	23
29	.4848	.4874	.4899	.4924	.4950	.4975	.5000	60°	3	5	8	10	13	15	18	20	23
30°	0.5000	0.5025	0.5050	0.5075	0.5100	0.5125	0.5150	59	3	5	8	10	13	15	18	20	23
31	.5150	.5175	.5200	.5225	.5250	.5275	.5299	58	2	5	7	10	12	15	17	20	22
32	.5299	.5324	.5348	.5373	.5398	.5422	.5446	57	2	5	7	10	12	15	17	20	22
33	.5446	.5471	.5495	.5519	.5544	.5568	.5592	56	2	5	7	10	12	15	17	19	22
34	.5592	.5616	.5640	.5664	.5688	.5712	.5736	55	2	5	7	10	12	14	17	19	22
35	0.5736	0.5760	0.5783	0.5807	0.5831	0.5854	0.5878	54	2	5	7	9	12	14	17	19	21
36	.5878	.5901	.5925	.5948	.5972	.5995	.6018	53	2	5	7	9	12	14	16	19	21
37	.6018	.6041	.6065	.6088	.6111	.6134	.6157	52	2	5	7	9	12	14	16	18	21
38	.6157	.6180	.6202	.6225	.6248	.6271	.6293	51	2	5	7	9	11	14	16	18	20
39	.6293	.6316	.6338	.6361	.6383	.6406	.6428	50°	2	4	7	9	11	13	16	18	20
40°	0.6428	0.6450	0.6472	0.6494	0.6517	0.6539	0.6561	49	2	4	7	9	11	13	15	18	20
41	.6561	.6583	.6604	.6626	.6648	.6670	.6691	48	2	4	7	9	11	13	15	17	20
42	.6691	.6713	.6734	.6756	.6777	.6799	.6820	47	2	4	6	9	11	13	15	17	19
43	.6820	.6841	.6862	.6848	.6905	.6926	.6947	46	2	4	6	8	11	13	15	17	19
44	.6947	.6967	.6988	.7009	.7030	.7050	.7071	45	2	4	6	8	10	12	15	17	19
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

ප්‍රකෘති කෝසයින
இயற்கைக் கோசைன்கள்
NATURAL COSINES

ප්‍රකෘති සයින
இயற்கைச் சைன்கள்
NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences 1′	2′	3′	4′	5′	6′	7′	8′	9′
45°	0.7071	0.7092	0.7112	0.7133	0.7153	0.7173	0.7193	44°	2	4	6	8	10	12	14	16	18
46	.7193	.7214	.7234	.7254	.7274	.7294	.7314	43	2	4	6	8	10	12	14	16	18
47	.7314	.7333	.7353	.7373	.7392	.7412	.7431	42	2	4	6	8	10	12	14	16	18
48	.7431	.7451	.7470	.7490	.7509	.7528	.7547	41	2	4	6	8	10	12	13	15	17
49	.7547	.7566	.7585	.7604	.7623	.7642	.7660	40°	2	4	6	8	9	11	13	15	17
50°	0.7660	0.7679	0.7698	0.7716	0.7735	0.7753	0.7771	39	2	4	6	7	9	11	13	15	17
51	.7771	.7790	.7808	.7826	.7844	.7862	.7880	38	2	4	5	7	9	11	13	14	16
52	.7880	.7898	.7916	.7934	.7951	.7969	.7986	37	2	4	5	7	9	11	12	14	16
53	.7986	.8004	.8021	.8039	.8056	.8073	.8090	36	2	3	5	7	9	10	12	14	16
54	.8090	.8107	.8124	.8141	.8158	.8175	.8192	35	2	3	5	7	8	10	12	14	15
55	0.8192	0.8208	0.8225	0.8241	0.8258	0.8274	0.8290	34	2	3	5	7	8	10	12	13	15
56	.8290	.8307	.8323	.8339	.8355	.8371	.8387	33	2	3	5	6	8	10	11	13	14
57	.8387	.8403	.8418	.8434	.8450	.8465	.8480	32	2	3	5	6	8	9	11	13	14
58	.8480	.8496	.8511	.8526	.8542	.8557	.8572	31	2	3	5	6	8	9	11	12	14
59	.8572	.8587	.8601	.8616	.8631	.8646	.8660	30°	1	3	4	6	7	9	10	12	13
60°	0.8660	0.8675	0.8689	0.8704	0.8718	0.8732	0.8746	29	1	3	4	6	7	9	10	11	13
61	.8746	.8760	.8774	.8788	.8802	.8816	.8829	28	1	3	4	6	7	8	10	11	12
62	.8829	.8843	.8857	.8870	.8884	.8897	.8910	27	1	3	4	5	7	8	9	11	12
63	.8910	.8923	.8936	.8949	.8962	.8975	.8988	26	1	3	4	5	6	8	9	10	12
64	.8988	.9001	.9013	.9026	.9038	.9051	.9063	25	1	3	4	5	6	8	9	10	11
65	0.9063	0.9075	0.9088	0.9100	0.9112	0.9124	0.9135	24	1	2	4	5	6	7	8	10	11
66	.9135	.9147	.9159	.9171	.9182	.9194	.9205	23	1	2	3	5	6	7	8	9	10
67	.9205	.9216	.9228	.9239	.9250	.9261	.9272	22	1	2	3	4	6	7	8	9	10
68	.9272	.9283	.9293	.9304	.9315	.9325	.9336	21	1	2	3	4	5	6	7	9	10
69	.9336	.9346	.9356	.9367	.9377	.9387	.9397	20	1	2	3	4	5	6	7	8	9
70°	0.9397	0.9407	0.9417	0.9426	0.9436	0.9446	0.9455	19	1	2	3	4	5	6	7	8	9
71	.9455	.9465	.9474	.9483	.9492	.9502	.9511	18	1	2	3	4	5	6	6	7	8
72	.9511	.9520	.9528	.9537	.9546	.9555	.9563	17	1	2	3	4	4	5	6	7	8
73	.9563	.9572	.9580	.9588	.9596	.9605	.9613	16	1	2	2	3	4	5	6	7	7
74	.9613	.9621	.9628	.9636	.9644	.9652	.9659	15	1	2	2	3	4	5	5	6	7
75	0.9659	0.9667	0.9674	0.9681	0.9689	0.9696	0.9703	14	1	1	2	3	4	4	5	6	7
76	.9703	.9710	.9717	.9724	.9730	.9737	.9744	13	1	1	2	3	3	4	5	5	6
77	.9744	.9750	.9757	.9763	.9769	.9775	.9781	12	1	1	2	3	3	4	4	5	6
78	.9781	.9787	.9793	.9799	.9805	.9811	.9816	11	1	1	2	2	3	3	4	5	5
79	.9816	.9822	.9827	.9833	.9838	.9843	.9848	10°	1	1	2	2	3	3	4	4	5
80°	0.9848	0.9853	0.9858	0.9863	0.9868	0.9872	0.9877	9	0	1	1	2	2	3	3	4	4
81	.9877	.9881	.9886	.9890	.9894	.9899	.9903	8	0	1	1	2	2	3	3	3	4
82	.9903	.9907	.9911	.9914	.9918	.9922	.9925	7	0	1	1	2	2	2	3	3	3
83	.9925	.9929	.9932	.9936	.9939	.9942	.9945	6	0	1	1	1	2	2	2	3	3
84	.9945	.9948	.9951	.9954	.9957	.9959	.9962	5	0	1	1	1	1	2	2	2	3
85	0.9962	0.9964	0.9967	0.9969	0.9971	0.9974	0.9976	4	(අන්තරය ඉතා කුඩා බැවින් වගු ගත කිරීම අනවශ්‍යය.)								
86	.9976	.9978	.9980	.9981	.9983	.9985	.9986	3									
87	.9986	.9988	.9989	.9990	.9992	.9993	.9994	2									
88	.9994	.9995	.9996	.9997	.9997	.9998	.9998	1									
89	0.9998	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	0°									
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

ප්‍රකෘති කෝසයින
இயற்கைக் கோசைன்கள்
NATURAL COSINES

ප්‍රකෘති ටැංජන

இயற்கைத் தான்கன்கள்

NATURAL TANGENTS

	0'	10'	20'	30'	40'	50'	60'		1'	2'	3'	4'	5'	6'	7'	8'	9'
									මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
0	0.0000	0.0029	0.0058	0.0087	0.0116	0.0145	0.0175	89°	3	6	9	12	15	17	20	23	26
1	.0175	.0204	.0233	.0262	.0291	.0320	.0349	88	3	6	9	12	15	17	20	23	26
2	.0349	.0378	.0407	.0437	.0466	.0495	.0524	87	3	6	9	12	15	18	20	23	26
3	.0524	.0553	.0582	.0612	.0641	.0670	.0699	86	3	6	9	12	15	18	20	23	26
4	.0699	.0729	.0758	.0787	.0816	.0846	.0875	85	3	6	9	12	15	18	21	23	26
5	0.0875	0.0904	0.0934	0.0963	0.0992	0.1022	0.1051	84	3	6	9	12	15	18	21	24	26
6	.1051	.1080	.1110	.1139	.1169	.1198	.1228	83	3	6	9	12	15	18	21	24	27
7	.1228	.1257	.1287	.1317	.1346	.1376	.1405	82	3	6	9	12	15	18	21	24	27
8	.1405	.1435	.1465	.1495	.1524	.1554	.1584	81	3	6	9	12	15	18	21	24	27
9	.1584	.1614	.1644	.1673	.1703	.1733	.1763	80°	3	6	9	12	15	18	21	24	27
10°	0.1763	0.1793	0.1823	0.1853	0.1883	0.1914	0.1944	79	3	6	9	12	15	18	21	24	27
11	.1944	.1974	.2004	.2035	.2065	0.2095	0.2126	78	3	6	9	12	15	18	21	24	27
12	.2126	.2156	.2186	.2217	.2247	.2278	.2309	77	3	6	9	12	15	18	21	24	27
13	.2309	.2339	.2370	.2401	.2432	.2462	.2493	76	3	6	9	12	15	18	22	25	28
14	.2493	.2524	.2555	.2586	.2617	.2648	.2679	75	3	6	9	12	16	19	22	25	28
15	0.2679	0.2711	0.2742	0.2773	0.2805	0.2836	0.2867	74	3	6	9	13	16	19	22	25	28
16	.2867	.2899	.2931	.2962	.2994	.3026	.3057	73	3	6	9	13	16	19	22	25	28
17	.3057	.3089	.3121	.3153	.3185	.3217	.3249	72	3	6	10	13	16	19	22	26	29
18	.3249	.3281	.3314	.3346	.3378	.3411	.3443	71	3	6	10	13	16	19	23	26	29
19	.3443	.3476	.3508	.3541	.3574	.3607	.3640	70°	3	7	10	13	16	20	23	26	29
20°	0.3640	0.3673	0.3706	0.3739	0.3772	0.3805	0.3839	69	3	7	10	13	17	20	23	27	30
21	.3839	.3872	.3906	.3939	.3973	.4006	.4040	68	3	7	10	13	17	20	24	27	30
22	.4040	.4074	.4108	.4142	.4176	.4210	.4245	67	3	7	10	14	17	20	24	27	31
23	.4245	.4279	.4314	.4348	.4383	.4417	.4452	66	3	7	10	14	17	21	24	28	31
24	.4452	.4487	.4522	.4557	.4592	.4628	.4663	65	4	7	11	14	18	21	25	28	32
25	0.4663	0.4699	0.4734	0.4770	0.4806	0.4841	0.4877	64	4	7	11	14	18	21	25	29	32
26	.4877	.4913	.4950	.4986	.5022	.5059	.5095	63	4	7	11	15	18	22	25	29	33
27	.5095	.5132	.5169	.5206	.5243	.5280	.5317	62	4	7	11	15	18	22	26	30	33
28	.5317	.5354	.5392	.5430	.5467	.5505	.5543	61	4	8	11	15	19	23	26	30	34
29	.5543	.5581	.5619	.5658	.5696	.5735	.5774	60°	4	8	12	15	19	23	27	31	35
30°	0.5774	0.5812	0.5851	0.5890	0.5930	0.5969	0.6009	59	4	8	12	16	20	24	27	31	35
31	.6009	.6048	.6088	.6128	.6168	.6208	.6249	58	4	8	12	16	20	24	28	32	36
32	.6249	.6289	.6330	.6371	.6412	.6453	.6494	57	4	8	12	16	20	25	29	33	37
33	.6494	.6536	.6577	.6619	.6661	.6703	.6745	56	4	8	13	17	21	25	29	33	38
34	.6745	.6787	.6830	.6873	.6916	.6959	.7002	55	4	9	13	17	21	26	30	34	39
35	0.7002	0.7046	0.7089	0.7133	0.7177	0.7221	0.7265	54	4	9	13	18	22	26	31	35	40
36	.7265	.7310	.7355	.7400	.7445	.7490	.7536	53	5	9	14	18	23	27	32	36	41
37	.7536	.7581	.7627	.7673	.7720	.7766	.7813	52	5	9	14	19	23	28	32	37	42
38	.7813	.7860	.7907	.7954	.8002	.8050	.8098	51	5	10	14	19	24	29	33	38	43
39	.8098	.8146	.8195	.8243	.8292	.8342	.8391	50°	5	10	15	20	24	29	34	39	44
40°	0.8391	0.8441	0.8491	0.8541	0.8591	0.8642	0.8693	49	5	10	15	20	25	30	35	40	45
41	.8693	.8744	.8796	.8847	.8899	.8952	.9004	48	5	10	16	21	26	31	36	41	47
42	.9004	.9057	.9110	.9163	.9217	.9271	.9325	47	5	11	16	21	27	32	37	43	48
43	.9325	.9380	.9435	.9490	.9545	.9601	.9657	46	6	11	17	22	28	33	39	44	50
44	.9657	.9731	.9770	.9827	.9884	.9942	1.0000	45	6	11	17	23	29	34	40	46	51
	60'	50'	40'	30'	20'	10'	0'		1'	2'	3'	4'	5'	6'	7'	8'	9'

ප්‍රකෘති කෝටැංජන

இயற்கைக் கோதான்சன்கள்

NATURAL COTANGENTS

ප්‍රකෘති ටැංජන
இயற்கைத் தான்சன்கள்
NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
45°	1.0000	1.0058	1.0117	1.0176	1.0235	1.0295	1.0355	44°	6	12	18	24	30	36	41	47	53
46	.0355	.0416	.0477	.0538	.0599	.0661	.0724	43	6	12	18	25	31	37	43	49	55
47	.0724	.0786	.0850	.0913	.0977	.1041	.1106	42	6	13	19	26	32	38	45	51	57
48	.1106	.1171	.1237	.1303	.1369	.1436	.1504	41	7	13	20	27	33	40	46	53	60
49	.1504	.1571	.1640	.1708	.1778	.1847	.1918	40°	7	14	21	28	34	41	48	55	62
50°	1.1918	1.1988	1.2059	1.2131	1.2203	1.2276	1.2349	39	7	14	22	29	36	43	50	58	65
51	.2349	.2423	.2497	.2572	.2647	.2723	.2799	38	8	15	23	30	38	45	53	60	68
52	.2799	.2876	.2954	.3032	.3111	.3190	.3270	37	8	16	24	31	39	47	55	63	71
53	.3270	.3351	.3432	.3514	.3597	.3680	.3764	36	8	16	25	33	41	49	58	66	74
54	.3764	.3848	.3934	.4019	.4106	.4193	.4281	35	9	17	26	35	43	52	60	69	78
55	1.4281	1.4370	1.4460	1.4550	1.4641	1.4733	1.4826	34	9	18	27	36	45	54	63	73	82
56	.4826	.4919	.5013	.5108	.5204	.5301	.5399	33	10	19	29	38	48	57	67	76	86
57	.5399	.5497	.5597	.5697	.5798	.5900	.6003	32	10	20	30	40	50	60	71	81	91
58	.6003	.6107	.6212	.6319	.6426	.6534	.6643	31	11	21	32	43	53	64	75	85	96
59	.6643	.6753	.6864	.6977	.7090	.7205	.7321	30°	11	23	34	45	56	68	79	90	102
60°	1.732	1.744	1.756	1.767	1.780	1.792	1.804	29	1	2	4	5	6	7	8	10	11
61	1.804	1.816	1.829	1.842	1.855	1.868	1.881	28	1	3	4	5	6	8	9	10	12
62	1.881	1.894	1.907	1.921	1.935	1.949	1.963	27	1	3	4	5	7	8	10	11	12
63	1.963	1.977	1.991	2.006	2.020	2.035	2.050	26	1	3	4	6	7	9	10	12	13
64	2.050	2.066	2.081	2.097	2.112	2.128	2.145	25	2	3	5	6	8	9	11	13	14
65	2.145	2.161	2.177	2.194	2.211	2.229	2.246	24	2	3	5	7	8	10	12	14	15
66	2.246	2.264	2.282	2.300	2.318	2.337	2.356	23	2	4	5	7	9	11	13	15	16
67	2.356	2.375	2.394	2.414	2.434	2.455	2.475	22	2	4	6	8	10	12	14	16	18
68	2.475	2.496	2.517	2.539	2.560	2.583	2.605	21	2	4	6	9	11	13	15	17	20
69	2.605	2.628	2.651	2.675	2.699	2.723	2.747	20°	2	5	7	9	12	14	17	19	21
70°	2.747	2.773	2.798	2.824	2.850	2.877	2.904	19	3	5	8	10	13	16	18	21	23
71	2.904	2.932	2.960	2.989	3.018	3.047	3.078	18	3	6	9	12	14	17	20	23	26
72	3.078	3.108	3.140	3.172	3.204	3.237	3.271	17	3	6	10	13	16	19	23	26	29
73	3.271	3.305	3.340	3.376	3.412	3.450	3.487	16	4	7	11	14	18	22	25	29	32
74	3.487	3.526	3.566	3.606	3.647	3.689	3.732	15	4	8	12	16	20	24	29	33	37
75	3.732	3.776	3.821	3.867	3.914	3.962	4.011	14	5	9	14	19	23	28	33	37	42
76	4.011	4.061	4.113	4.165	4.219	4.275	4.331	13	5	11	16	21	27	32	37	43	48
77	4.331	4.390	4.449	4.511	4.574	4.638	4.705	12	6	12	19	25	31	37	44	50	56
78	4.705	4.773	4.843	4.915	4.989	5.066	5.145	11	7	15	22	29	37	44	51	59	66
79	5.145	5.226	5.309	5.396	5.485	5.576	5.671	10°	9	18	26	35	44	53	61	70	79
80°	5.671	5.769	5.871	5.976	6.084	6.197	6.314	9									
81	6.314	6.435	6.561	6.691	6.827	6.968	7.115	8									
82	7.115	7.269	7.429	7.596	7.770	7.953	8.144	7	අන්තර ශීඝ්‍රයෙන් වෙනස් වෙයි								
83	8.144	8.345	8.556	8.777	9.010	9.255	9.514	6									
84	9.514	9.788	10.078	10.385	10.712	11.059	11.430	5	வித்தியாசங்கள் விரைந்து மாறுவன								
85	11.43	11.83	12.25	12.71	13.20	13.73	14.30	4									
86	14.30	14.92	15.60	16.35	17.17	18.07	19.08	3	Differences change rapidly								
87	19.08	20.21	21.47	22.90	24.54	26.43	28.64	2									
88	28.64	31.24	34.37	38.19	42.96	49.10	57.29	1									
89	57.29	68.75	85.94	114.59	171.89	343.77	∞	0°									
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

ප්‍රකෘති කෝටැංජන
இயற்கைக் கோதான்சன்கள்
NATURAL COTANGENTS

Glossary

A

Adjacent side	බද්ධ පාදය	அயற் பக்கம்
Angles in the same segment	ඒකාන්තර වෘත්ත ඛණ්ඩයේ කෝණ	ஒரேதுண்டக் கோணங்கள்

C

Centre	කේන්ද්‍රය	மையம்
Chord	ජ්‍යාය	நாண்
Circle	වෘත්තය	வட்டம்
Circumcircle	පරිවෘත්තය	சுற்று வட்டம்
Column matrix	තීර න්‍යාසය	நிரற் தாயம்
Cosine	කෝසයිනය	கோசைன்
Cyclic quadrilateral	වෘත්ත චතුරස්‍ර	வட்ட நாற்பக்கல்

D

Dependent Events	පරායත්ත සිද්ධි	சார் நிகழ்ச்சி

E

Element	අවයව	மூலகம்
Elements of a matrix	න්‍යාසයක අවයව	தாயமொன்றின் மூலகங்கள்
Exterior angle	බාහිර කෝණය	புறக்கோணம்
Exterior point	බාහිර ලක්ෂ්‍යය	புறப்புள்ளி
Excircle/Escribed circle	බාහිර් වෘත්තය	வெளி வட்டம்

G

Grid	කොටු දැල	நெய்யரி

H

Hypotenuse	කර්ණය	செம்பக்கம்

I

Independent events	ස්වායත්ත සිද්ධි	சாரா நிகழ்ச்சிகள்
Inequalities	අසමානතා	சமனிலிகள்
Inscribed circle/ Incircle	අන්තර්වෘත්තය	உள்வட்டம்
Interior opposite angle	අභ්‍යන්තර සම්මුඛ කෝණය	அகத்தெதிர் கோணம்
Intersection of sets	කුලක ඡේදනය	தொடைகளின் இடைவெட்டு

L		
Locus	පථය	ஒழுக்கு
M		
Matrices	න්‍යාස	தாயங்கள்
O		
Opposite angles	සම්මුඛ කෝණ	எதிர்க் கோணங்கள்
Opposite side	සම්මුඛ පාදය	எதிர்ப் பக்கங்கள்
Order of a matrix	න්‍යාසයේ ගණය	தாயத்தின் வரிசை
P		
Perpendicular	ලම්බකය	செங்குத்து
Point	ලක්ෂ්‍යය	புள்ளி
Pythagoras' theorem	පයිතගරස් ප්‍රමේයය	பைதகரசின் தேற்றம்
Pythagoras' triple	පයිතගරස් ත්‍රික	பைதகரசின் மும்மை
R		
Radius	අරය	ஆரை
Random Experiments	සසම්භාවී පරීක්ෂණ	எழுமாற்றுப் பரிசோதனை
Riders	අනුමේයයන්	ஏறிகள்
Right angled triangles	සෘජුකෝණික ත්‍රිකෝණ	செங்கோண முக்கோணம்
Row matrix	පේළි න්‍යාසය	நிரைத் தாயம்
S		
Sample space	නියැදි අවකාශය	மாதிரிவெளி
Segment of a circle	වෘත්ත ඛණ්ඩය	வட்டத்துத்துண்டம்
Set	කුලකය	தொடை
Sine	සයිනය	சைன்
Solution set	විසඳුම් කුලකය	தீர்வுத் தொடை
Square matrix	සමචතුරස්‍ර න්‍යාසය	சதுரத் தாயம்
Subtended	ආපාතිත	எதிரமை
Supplementary	පරිපූරක	மிகை நிரப்புகின்ற
Symmetric matrix	සමමිති න්‍යාසය	சமச்சீர்த் தாயம்

T

Tangent	ස්පර්ශකය	தொடலி
Tree diagram	රුක් සටහන	மரவரிப்படம்
Trigonometric ratios	ත්‍රිකෝණමිතික අනුපාත	திரிகோண கணித விகிதங்கள்
Trignometry	ත්‍රිකෝණමිතිය	திரிகோண கணிதம்

U

Union of sets	කුලක මේලය	தொடைகளின் ஒன்றிப்பு
Unit matrix	ඒකක න්‍යාසය	அலகுத் தாயம்

V

Venn diagram	වෙන් රූපය	வென் வரிப்படம்

Sequence of the Lessons

Chapter of Textbook	No.of Periods
1 Term	
1. Real Numbers	10
2. Indices and Logarithms I	08
3. Indices and Logarithms II	06
4. Surface Area of Solids	05
5. Volume of the Solids	05
6. Binomial Expressions	04
7. Algebraic Fractions	04
8. Areas of Plane Figures between Parallel Lines	12
2 Term	
09. Percentages	06
10. Share Market	05
11. Mid Point Theorem	05
12. Graphs	12
13. Formulae	10
14. Equiangular Triangles	12
15. Data representaion and Interpretation	12
16. Geometric Progressions	06
3 Term	
17. Pythagoras's Theorem	04
18. Trignometry	12
19. Matrices	08
20. Inequalities	06
21. Cyclic Quadrilaterals	10
22. Tangent	10
23. Constructions	05
24. Sets	06
25. Probability	07