MATHEMATICS

Grade 11

Part - I

Educational Publications Department

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The National Anthem of Sri Lanka

Sri Lanka Matha
Apa Sri Lanka Namo Namo Namo Namo Matha
Sundara siri barinee, surendi athi sobamana Lanka
Dhanya dhanaya neka mal palaturu piri jaya bhoomiya ramya
Apa hata sepa siri setha sadana jeewanaye matha
Piliganu mena apa bhakthi pooja Namo Namo Matha
Apa Sri Lanka Namo Namo Namo Namo Matha
Oba we apa vidya
Obamaya apa sathya
Oba we apa shakthi
Apa hada thula bhakthi
Oba apa aloke
Apage anuprane
Oba apa jeevana we
Apa mukthiya oba we
Nava jeevana demine, nithina apa pubudukaran matha
Gnana veerya vadawamina regena yanu mana jaya bhoomi kara
Eka mavakage daru kela bevina
Yamu yamu vee nopama
Prema vada sema bheda durerada
Namo, Namo Matha
Apa Sri Lanka Namo Namo Namo Namo Matha

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ආනන්ද සමරකෝන්

ஒரு தாய் மக்கள் நாமாவோம்
ஒன்றே நாம் வாழுமில்லம்
நன்றே உடலில் ஓடும்
ஒன்றே நம் குருதி நிறம்

அதனால் சகோதரர் நாமாவோம்
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கவிதையின் பெயர்ப்பு

Foreword

With the continuous advancement of the world, the education sector too is transformed. Therefore, if we require the creation of a student community who could confront the future challenges successfully, our learning teaching process must constantly utilize effective approaches. It is our responsibility to disseminate the knowledge of the new world while assisting to create global citizens with good values. Our department is actively engaged in producing learning tools with the great aim of contributing to enlighten the minds of the children of the country.

A textbook is a repository of knowledge. At times, it takes us to a world of entertainment while developing our critical thinking faculties. It promotes our hidden potentials. In the coming years, the memories related to these textbooks will bring you happiness. While making the maximum use of this valuable learning tool, you must essentially access other useful knowledge spaces too. This textbook is offered to you free of charge as a great gift of the free education. Only you can add a value to the great fortune that has been spent by the government to print these textbooks. I wish that you would gain the ability to enlighten the future as citizens with knowledge and values by using this textbook.

I would like to bestow my sincere thanks on the panels of writers, editors and reviewers as well as on the staff of the Educational Publications Department for the contribution made on this endeavor.

P. N. Ilapperuma,
Commissioner General of Educational Publications,
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26. 06. 2020

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Message of the Board of Compilers

This textbook has been compiled in accordance with the new syllabus to be implemented from 2015.

Textbooks are compiled for students. Therefore, we have made an attempt to compile this textbook in a simple and detailed manner making it possible for you to read and understand it on your own.

We have included descriptions, activities and examples to introduce the subject concepts in an attractive manner and to establish them. Moreover, activities are organized from simple to complex to develop an interest to do them.

We have used the terms related to mathematical concepts in accordance with the glossary of technical terms of mathematics compiled by the Department of Official Languages.

Some subject matter learnt during the earlier grades is necessary to learn the subject content in the grade 11 syllabus. Thus, review exercises are included at the beginning of each chapter to revise previous knowledge. You will be prepared by them for the subject content of grade 11.

In addition, students may use the grade 10 book which you have if you need to recall previuos knoweledge.

You will gain maximum benefit from this textbook by reading the chapters and doing the review exercises of each chapter even before your teacher teaches them in the classroom.

We hope that studying mathematics will be an interesting, joyful and productive experience.

Board of Compilers

1 Real Numbers

By studying this lesson will be able to

- to investigate number sets
- workout basic mathematical operations regarding surds.

It is believed that the concept of numbers originated among the human race about 30 000 years ago. This concept which originated and developed independently in various civilizations, evolved globally and has now become a universal field of study named mathematics.

It can be assumed that numbers were initially used in early civilizations for simple purposes such as counting and accounting. There is no doubt that the first numerical concepts that were developed were "one" and "two". Later the concepts of three, four etc., must have been developed. Then man would have realized that he could name any amount that he wished in this manner. Different civilizations used different symbols to name numbers.

It is accepted based on historical evidence, that the numerals 1, 2, 3 etc., which we now use, originated in India. The honour of being the first to use the concept of zero as a number as well as being the first to introduce the positional decimal number system also goes to India. This number system is now defined as the Hindu-Arabic number system and the modern belief is that it was first taken to the Middle-East and then to Europe by traders. This system is the standard number system which is accepted and used worldwide now.

The manipulation of numbers using the basic mathematical operations (addition, subtraction, multiplication and division) can be considered as a great revolution in the history of mankind in relation to the use of numbers. In this age of technology it is unimaginable to think of the existence of man without numbers and the operations performed on them.

Although the numbers 1, 2, 3 etc., can be considered as the first numbers that were used to fulfill certain needs of man, later the number zero, fractional numbers and negative numbers were also included in the number system. During the period when mathematics was developing as a separate field, the attention of mathematicians was directed towards various other types of numbers (sets) too. In this lesson we hope to study about such sets of numbers, their notations and properties.

The Set of Integers ($\mathbb{Z}$)

It is natural that we identify initially the numbers 1, 2, 3, … which we first learnt about as children. These numbers are defined as **counting numbers** and the set which consists of all these numbers is written using set notation as follows.

$$\{1, 2, 3, \ldots\}$$

The reason for this set of numbers to be called the counting numbers is very clear. However its mathematical usage in modern times is limited. The name used most often now for this set is "**the set of positive integers**". This set is denoted by $\mathbb{Z}^+$.

$$\text{Thus, } \mathbb{Z}^+ = \{1, 2, 3, \ldots\}.$$

That is, the numbers 1, 2, 3, … are called positive integers.

The numbers defined as negative integers are -1, -2, -3, etc. Although there is no commonly used symbol to denote this set, some mathematicians, based on the needs of their field of study, use the symbol $\mathbb{Z}^-$.

The positive integers, zero and the negative integers together form the set of **integers**. This set is denoted by $\mathbb{Z}$. Accordingly,

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

or equivalently,

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

The Set of Natural Numbers ($\mathbb{N}$)

Let us consider again the set of numbers 1, 2, 3,... This set is also defined as the set of **natural numbers** and is denoted by $\mathbb{N}$.

$$\text{That is, } \mathbb{N} = \{1, 2, \ldots\}.$$

Note: There is no consensus among mathematicians regarding which numbers should be considered as natural numbers. The suitability of calling the numbers 1,2,3, … natural numbers is clear. However some of the mathematicians (especially specialists in set theory), have considered 0 as a natural number in their books.

One reason may be because at that time there was no accepted name nor accepted symbol for the set consisting of 0 and the positive integers.However most books on number theory consider the set of natural numbers to be the set $\{1, 2, 3, ...\}$.Almost all authors of mathematics books now mention at the beginning of their books which set of numbers they consider as the natural numbers.

The set of Rational Numbers ($\mathbb{Q}$)

We have come across earlier that, like the integers, fractions too can be considered as numbers, and that operations such as addition and multiplication can be performed on them too. Every integer can be written as a fraction. (For example, we can write $2 = \frac{2}{1}$). Further, a fraction can be written in different forms, all having the same numerical value. (For example, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$). We have also come across negative fractions ($-\frac{2}{5}, -\frac{11}{3}$, etc.). Although we usually think that the numerator and denominator of a fraction should consist of integers, this is actually not the case. For example, $\frac{3}{\sqrt{2}}$ is also a fraction. However, fractions with integers in both the numerator and the denominator (apart from 0 in the denominator), have an important place in mathematics. They are called **rational numbers**. The set of rational numbers is denoted by $\mathbb{Q}$. Accordingly, the set of rational numbers can be defined using the set builder notation as follows.

$$\mathbb{Q} = \left\{ \frac{a}{b} \ : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$

There are other ways too of defining the set of rational numbers. One other way is as follows.

$$\mathbb{Q} = \left\{ \frac{a}{b} \ : a \in \mathbb{Z} \, , b \in \mathbb{Z}^{+} \right\}.$$

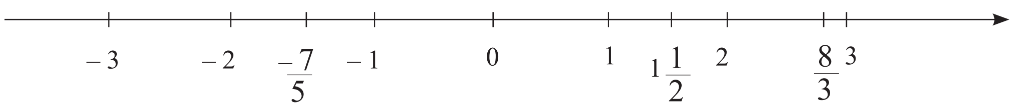
Both these definitions are equivalent. Since the denominator of a rational number cannot be 0 and since all the negative rational numbers can be obtained by considering the fractions with the negative integers in the numerator and positive integers in the denominator.

The Set of Irrational Numbers $(\mathbb{Q}')$

It is appropriate to define the irrational numbers now. Do you recall how you learnt about numbers in previous grades by drawing a number line? Let us reconsider this now.

Let us consider a straight line which can be lengthened as required in either direction. Let us name a point we like on that line as the origin 0. Let us assume that we have marked all the numbers 1, 2, 3, etc., on one side of 0 (usually the right hand side) and all the numbers – 1, – 2, – 3 etc., on the opposite side, keeping equal gaps between the numbers. That is, let us assume that the points corresponding to all the integers have been marked on this number line. Let us also assume that the points corresponding to all the rational numbers too have been marked on this line.

The figure below shows several such points that have been marked.



Accordingly, all the rational numbers (including the integers) are now assumed to have been marked on this line. Now, do you think that corresponding to each point on the line, a number has been marked? If asked differently, do you think that the distance from 0 to each point on the line can be written as a rational number? In truth, there are several points remaining on the number line which have not been marked. That is, there are points remaining on this number line that cannot be represented by a rational number. It is clear that the points that are remaining are those which correspond to the numbers which **cannot** be written in the form $\frac{a}{b}$, where a and b are integers. The numbers which correspond to the remaining points are defined as irrational numbers.

There is no specific symbol to denote the set of irrational numbers and it is usually denoted by $\mathbb{Q}'$, the complement of the set $\mathbb{Q}$. The numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ can be given as examples of irrational numbers. In fact, the square root of any positive integer which is not a perfect square is an irrational number. Apart from these, mathematicians have proved that π, which is the ratio of the circumference of any circle to its diameter, is also an irrational number. We take the value of π to be $\frac{22}{7}$ as an approximate value, for the convenience of performing calculations.

The Set of Real Numbers ($\mathbb{R}$)

According to the above discussion, all the numbers on a number line can be represented by rational numbers or irrational numbers. We call all the rational numbers together with all the irrational numbers, that is, all the numbers that can be represented on a number line, the **real numbers**. The set of real numbers is denoted by $\mathbb{R}$.

The Decimal Representation of a Number

Any real number can be represented as a decimal number. Initially, let us consider the decimal representation of several rational numbers.

1. The decimal representation of a rational number

$$4 = 4.000\ \ldots$$

$$\frac{1}{2} = 0.5 = 0.5000\ \ldots$$

$$\frac{11}{8} = 1.375 = 1.375000\ \ldots$$

$$\frac{211}{99} = 2.131313\ldots$$

$$\frac{767}{150} = 5.11333\ldots$$

$$\frac{37}{7} = 5.285714285714285714\ \ldots$$

A common property of these decimal representations is that starting at a certain point to the right of the decimal point (or from the beginning), one set of numerals (or one numeral) is recurring. Recurring means that it keeps repeating itself.

For example, in the decimal representation of $\frac{1}{2}$, the numeral 0 recurs starting from the second decimal place. The numeral 0 recurs from the first decimal place in the decimal representation of 4, the pair of numerals 13 recurs from the beginning in the decimal representation of $\frac{211}{99}$ and the group of numerals 285714 recurs from the beginning in the decimal representation of $\frac{37}{7}$. This property, that is, a group of numerals recurring continuously, is a property common to all rational numbers. If the portion that recurs is just 0, such a decimal representation is defined as a **finite decimal** (or **terminating decimal**). The decimals of which the portion that recurs is not zero, are called **recurring decimals**. Accordingly, $\frac{1}{2}$, 4 and $\frac{11}{8}$ in the above example are finite decimals while the rest are recurring decimals.

The above discussion leads to the following statement.

Every rational number can be written as a finite decimal or a recurring decimal.

Let us now learn a marvelous result regarding rational numbers. Suppose the rational number $\frac{a}{b}$ has a finite decimal representation. Let us assume that a and b have no common factors. Then the denominator (that is, b) has only powers of 2 or 5 (or both) as its factors. A rational number which has a recurring decimal representation must have a prime factor other than 2 and 5 in its denominator.

Recurring decimals are written in a concise form, by placing a dot above a numeral or numerals as shown in the following examples to indicate that they are recurring.

Recurring Decimal	Written Concisely
12.4444	$12.\dot{4}$
2.131313...	$2.\dot{1}\dot{3}$
5.11333...	$5.11\dot{3}$
5.285714285714285714...	$5.\dot{2}8571\dot{4}$

Exercise 1.1

1. For each of the following rational numbers state whether it is a finite decimal or a recurring decimal. Express the fractions which are recurring decimals in decimal form and then write them in a concise form.

a. $\frac{3}{4}$ **b.** $\frac{5}{5}$ **c.** $\frac{3}{7}$ **d.** $\frac{5}{9}$ **e.** $\frac{5}{21}$ **f.** $\frac{7}{32}$

g. $\frac{19}{33}$ **h.** $\frac{13}{50}$ **i.** $\frac{7}{64}$ **j.** $\frac{5}{18}$ **k.** $\frac{15}{128}$ **l.** $\frac{41}{360}$

2. The decimal representation of an irrational number

Finally, let us consider the decimal representation of an irrational number. In the decimal representation of an irrational number, no group of numerals recurs. For example, when the decimal representation of $\sqrt{2}$ is written down up to 60 decimal places, we obtain the following:

1.414213562373095048801688724209698078569671875376948073176679

π, which is a number we come across often is also an irrational number. When the value of π is calculated up to 60 decimal places we obtain the following:

3.141592653589793238462643383279502884197169399375105820974944

The following statements can be made regarding irrational numbers.

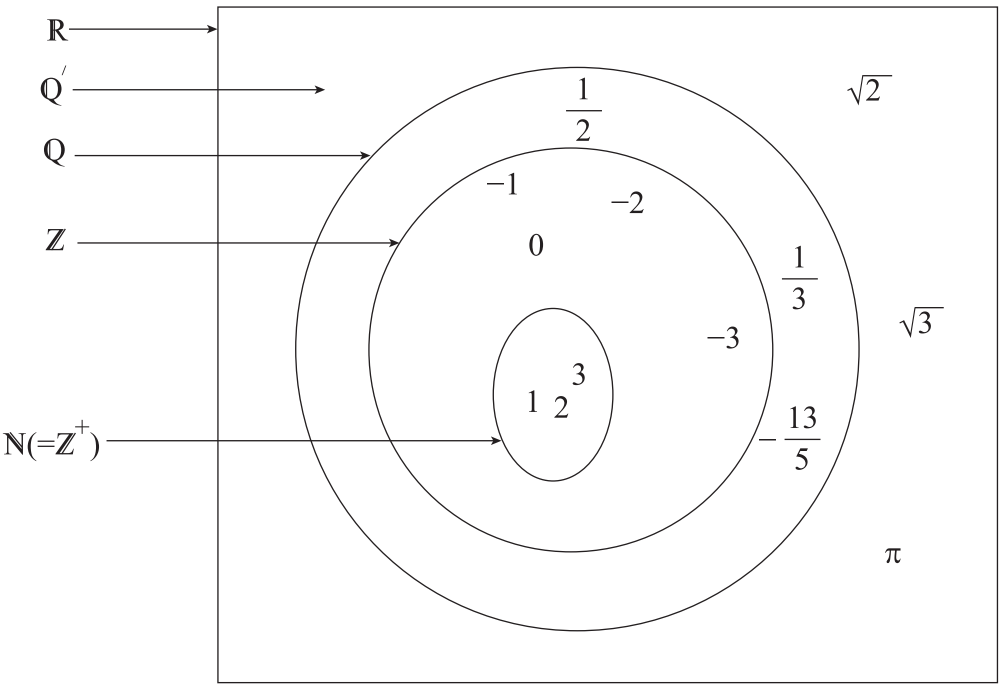
There is no group of recurring numerals in the decimal representation of an irrational number. If the decimal representation of a number is not finite, it is called **an infinite decimal**. Accordingly rational numbers which are recurring decimals and irrational numbers have infinite decimal representations.

Note: When describing the decimal representation of irrational numbers, one frequent error that is made is stating that "there is no pattern in the decimal representation of an irrational number". The issue here is that the word "pattern" is not well defined in mathematics. For example, the following decimal number has a very clear pattern.

0.101001000100001000001...

However, this is an irrational number, Observe that there is no group of recurring numbers.

All the sets of numbers that have been studied so far can be represented in a Venn diagram as follows, with the set of real numbers as the universal set, and the other sets as its subsets. A few numbers belonging to each subset have been included to make it easier for you to understand the relationship between the sets.



Exercise 1.2

1. State whether the following real numbers as rational numbers or irrational numbers.

a. $\sqrt{2}$ **b.** $\sqrt{25}$ **c.** $\sqrt{6}$ **d.** $\sqrt{11}$ **e.** 6.52

2. Determine whether each of the following statement is true or false.

a. Any real number is a finite decimal or an infinite decimal.
b. There can be rational numbers with infinite decimals representations.
c. Any real number is a recurring decimal or an infinite decimal.
d. 0.010110111011110... is a rational number.

1.2 Surds

There is no doubt that you recall how numerical (and algebraic) expressions are written using the symbol " $\sqrt{}$ " which is defined as the radical sign. For example, $\sqrt{4}$ is defined as the **positive square root** of 4 and it represents the positive number which when squared is equal to 4; that is 2. The positive square root is referred to as square root too. If a certain positive integer x is such that its square root $\sqrt{x}$ is also a positive integer, then x is defined as a perfect square. Accordingly, 4 is a perfect square. Since $\sqrt{4}$ is equal to 2. However, $\sqrt{2}$ is not the square root of a perfect square. We observed earlier that $\sqrt{2}$ is approximately equal to 1.414. We also learnt earlier in this lesson that $\sqrt{2}$ is an irrational number. A numerical term involving the symbol $\sqrt{}$ of which the value is not a rational number is defined as a surd.

The radical sign "$\sqrt{}$ " is used not only to denote the square roots of numbers, but also other roots. For example, $\sqrt[3]{2}$ denotes the positive number which when raised to the power 3 is equal to 2. This is called the cube root of 2. This is also an irrational number. Its value is approximately 1.2599. (You can verify this by finding the value of 1.2599^3). We can define the fourth root of 2 and the fifth root of 2 etc., in a similar manner. Such definitions can be made for other positive numbers as well; for example, $\sqrt[3]{5}$ and $\sqrt[6]{8.24}$. Such expressions are also surds. However, in this lesson we will only consider surds which are square roots of positive integers.

The square root of a positive integer which is not a perfect square is neither a finite decimal nor a recurring decimal. Observe that surds are therefore always irrational numbers.

Here our focus is on simplifying expressions which involve surds. There are many reasons why such simplifications are important. One reason is that it facilitates calculations. For example, when it is necessary to find the value of $\frac{1}{\sqrt{2}}$, if we take $\sqrt{2}$ to be approximately equal to 1.414, we would need to find the value of $\frac{1}{1.414}$. This division is fairly long. However, by simplifying this in the following manner, calculations are made easier.

$$\frac{1}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \text{ (Multiplying the numerator and denominator by } \sqrt{2}\text{)}$$

$$= \frac{\sqrt{2}}{2}$$

$$= \frac{1.414}{2} = 0.707.$$

Another reason for simplifying surds is to minimize errors during calculations.

For example, let us find the value of $\frac{\sqrt{20}}{2} - \sqrt{5}$. Let us use 4.5 as an approximate value for $\sqrt{20}$, and 2.2 as an approximate value for $\sqrt{5}$. Then

$$\begin{aligned}\frac{\sqrt{20}}{2} - \sqrt{5} &= \frac{4.5}{2} - 2.2 \\ &= 2.25 - 2.2 \\ &= 0.05\end{aligned}$$

However, the actual value of this expression is 0. One reason for getting a different answer is because we used approximate values for $\sqrt{20}$ and $\sqrt{5}$. However, by simplifying the above expression in a different way, we can get the correct value which is 0.

Surds appear in various form.

A special feature of a surd of the form $\sqrt{20}$ is that the whole number is under the radical sign. Such surds are defined as **entire surds**. $6\sqrt{15}$ means $6 \times \sqrt{15}$. This is a product of a surd and a rational number (not equal to 1). This is not an entire surd.
A surd is in its simplest form when it is in the form $a\sqrt{b}$; where a is a rational number and b has no factors which are perfect squares. For example $6\sqrt{15}$ is in its simplest form but $5\sqrt{12}$ is not, since 4 which is a perfect square is a factor of 12.

Now let us consider how to simplify expressions that contain surds of various forms.

Example 1

Simplify $3\sqrt{5} + 6\sqrt{5}$.

This can be simplified by considering $\sqrt{5}$ to be an unknown term.

$$3\sqrt{5} + 6\sqrt{5} = 9\sqrt{5}.$$

This simplification is similar to the simplification $3x + 6x = 9x$.

Observe that the expression obtained above in surd form cannot be simplified further. Keep in mind that simplifying further by using an approximate value for $\sqrt{5}$ is not what is meant by simplifying surds.
You should also keep in mind the important fact that an expression of the form $3\sqrt{2} + 8\sqrt{3}$ cannot be simplified further.

Now let us through examples, consider how expressions with surds are simplified by applying the properties of indices.

Example 2

Express the entire surd $\sqrt{20}$ as a surd.

$$\begin{aligned}\sqrt{20} &= \sqrt{4\times 5} \\ &= \sqrt{4}\times\sqrt{5} \qquad (\text{Since } \sqrt{ab}=\sqrt{a}\times\sqrt{b}) \\ &= 2\times\sqrt{5} \\ &= \underline{\underline{2\sqrt{5}}}\end{aligned}$$

Example 3

Express the surd $4\sqrt{5}$ as an entire surd.

$$\begin{aligned}4\sqrt{5} &= \sqrt{16}\times\sqrt{5} \qquad (\text{Since } 4=\sqrt{16}) \\ &= \sqrt{16\times 5} \\ &= \underline{\underline{\sqrt{80}}}\end{aligned}$$

Next let us consider how multiplication and division are performed on surds.

Example 4

Simplify: $5\sqrt{3}\times 4\sqrt{2}$.

Let us multiply the rational and irrational parts separately.

$$\begin{aligned}5\sqrt{3}\times 4\sqrt{2} &= 5\times 4\times\sqrt{3}\times\sqrt{2} \\ &= 20\times\sqrt{3\times 2} \\ &= \underline{\underline{20\sqrt{6}}}\end{aligned}$$

Example 5

Simplify: $3\sqrt{20}\div 2\sqrt{5}$.

The surd $3\sqrt{20}$ can be written as $3\sqrt{4\times 5}$. Simplifying further, it can be written as $3\times 2\sqrt{5} = 6\sqrt{5}$

$$\therefore\ 3\sqrt{20}\div 2\sqrt{5} = \frac{3\sqrt{20}}{2\sqrt{5}} = \frac{6\sqrt{5}}{2\sqrt{5}}$$

$$= \underline{\underline{3}}$$

Next we will consider how expressions of the form $\frac{a}{\sqrt{b}}$ are simplified. Examples for such expressions are $\frac{3}{\sqrt{2}}$ and $\frac{4}{\sqrt{5}}$. Expression of this form has a square root term in the denominator.

Now let us consider how such an expression can be converted into an expression with an integer (or a rational number) in the denominator.

Example 6

Express $\frac{3}{\sqrt{2}}$ as a fraction with an integer in the denominator.

The method used here is, to multiply both the numerator and the denominator of $\frac{3}{\sqrt{2}}$ by $\sqrt{2}$

$$\frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \underline{\underline{\frac{3\sqrt{2}}{2}}}$$

This process is defined as **rationalizing the denominator**.

Example 7

Rationalise the denominator of $\frac{a}{\sqrt{b}}$

$$\frac{a}{\sqrt{b}} = \frac{a \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}}$$
$$= \underline{\underline{\frac{a\sqrt{b}}{b}}}$$

Now let us consider how an expression involving surds is simplified.

Example 8

Simplify $4\sqrt{63} - 5\sqrt{7} - 8\sqrt{28}$.

$$4\sqrt{63} = 4 \times \sqrt{9 \times 7} = 4 \times 3\sqrt{7}$$
$$= 12\sqrt{7}$$
$$8\sqrt{28} = 8 \times \sqrt{4 \times 7} = 8 \times 2\sqrt{7}$$
$$= 16\sqrt{7}$$

Therefore $4\sqrt{63} - 5\sqrt{7} - 8\sqrt{28} = 12\sqrt{7} - 5\sqrt{7} - 16\sqrt{7}$
$$= \underline{\underline{-9\sqrt{7}}}$$

Let us consider how a more complex expression involving surds is simplified.

Example 9

Simplify $\frac{2\sqrt{6}}{\sqrt{2}} + \sqrt{75} - \frac{3}{\sqrt{12}}$

$$\frac{2\sqrt{6}}{\sqrt{2}} + \sqrt{75} - \frac{3}{\sqrt{12}} = \frac{2\sqrt{2 \times 3}}{\sqrt{2}} + \sqrt{25\times 3} - \frac{3}{\sqrt{4 \times 3}}$$

$$= \frac{2\sqrt{2} \times \sqrt{3}}{\sqrt{2}} + \sqrt{25\times 3} - \frac{3}{\sqrt{4 \times 3}}$$

$$= 2\sqrt{3} + 5\sqrt{3} - \frac{3}{2\sqrt{3}}$$

$$= 7\sqrt{3} - \frac{3 \times \sqrt{3}}{2\sqrt{3} \times \sqrt{3}}$$

$$= 7\sqrt{3} - \frac{3\sqrt{3}}{2 \times 3}$$

$$= 7\sqrt{3} - \frac{\sqrt{3}}{2}$$

$$= \frac{13\sqrt{3}}{2}$$

Exercise 1.3

1. Convert the following entire surds into surds.

a. $\sqrt{20}$ **b.** $\sqrt{48}$ **c.** $\sqrt{72}$ **d.** $\sqrt{28}$

e. $\sqrt{80}$ **f.** $\sqrt{45}$ **g.** $\sqrt{75}$ **h.** $\sqrt{147}$

2. Convert the following surds into entire surds.

a. $2\sqrt{3}$ **b.** $2\sqrt{5}$ **c.** $4\sqrt{7}$ **d.** $5\sqrt{2}$ **e.** $6\sqrt{11}$

3. Simplify.

a. $\sqrt{2} + 5\sqrt{2} - 2\sqrt{2}$

b. $\sqrt{5} + 2\sqrt{7} + 2\sqrt{5} - 3\sqrt{7}$

c. $4\sqrt{3} + 5\sqrt{2} + 3\sqrt{5} - 3\sqrt{2} + 3\sqrt{5} - 2\sqrt{3}$

d. $6\sqrt{11} + 3\sqrt{7} - 2\sqrt{11} - 5\sqrt{7} + 4\sqrt{7}$

e. $8\sqrt{3} + 7\sqrt{7} - 2\sqrt{3} + 3\sqrt{7} - 3\sqrt{7}$

4. Rationalise the denominator of the following fractions.

a. $\dfrac{2}{\sqrt{5}}$ **b.** $\dfrac{5}{\sqrt{3}}$ **c.** $\dfrac{5}{\sqrt{7}}$ **d.** $\dfrac{12}{2\sqrt{3}}$ **e.** $\dfrac{27}{3\sqrt{2}}$

f. $\dfrac{3}{2\sqrt{5}}$ **g.** $\dfrac{3\sqrt{5}}{2\sqrt{7}}$ **h.** $\dfrac{2\sqrt{3}}{3\sqrt{2}}$ **i.** $\dfrac{3\sqrt{3}}{2\sqrt{5}}$

5. Simplify.

a. $3\sqrt{2} \times 2\sqrt{3}$ **b.** $5\sqrt{11} \times 3\sqrt{7}$ **c.** $\sqrt{5} \times 3\sqrt{3}$

d. $4\sqrt{7} \div 2\sqrt{14}$ **e.** $6\sqrt{27} \div 3\sqrt{3}$ **f.** $\sqrt{48} \div 5\sqrt{3}$

6. Simplify.

a. $2\sqrt{27} - 3\sqrt{3} + 4\sqrt{7} + 3\sqrt{28}$ **b.** $3\sqrt{63} - 2\sqrt{7} + 3\sqrt{27} + 3\sqrt{3}$

c. $2\sqrt{128} - 3\sqrt{50} + 2\sqrt{162} + \dfrac{4}{\sqrt{2}}$ **d.** $\sqrt{99} - 2\sqrt{44} + \dfrac{110}{\sqrt{11}}$

e. $\dfrac{\sqrt{20}}{2} - \sqrt{5}$

2

Indices and Logarithms I

By studying this lesson, you will be able to

- simplify expressions involving powers and roots and
- solve equations

using the laws of indices and logarithms.

Indices

Do the following exercise to revise what you have learned so far about indices and logarithms.

Review Exercise

1. Simplify and find the value.

a. $2^2 \times 2^3$ **b.** $(2^4)^2$ **c.** 3^{-2}

d. $\dfrac{5^3 \times 5^2}{5^5}$ **e.** $\dfrac{3^5 \times 3^2}{3^6}$ **f.** $(5^2)^2 \div 5^3$

g. $\dfrac{(2^2)^3 \times 2^4}{2^8}$ **h.** $\dfrac{5^{-3} \times 5^2}{5^0}$ **i.** $(5^2)^{-2} \times 5 \times 3^0$

2. Simplify.

a. $a^2 \times a^3 \times a$ **b.** $a^5 \times a \times a^0$ **c.** $(a^2)^3$

d. $(x^2)^3 \times x^2$ **e.** $(xy)^2 \times x^0$ **f.** $(2x^2)^3$

g. $\dfrac{2pq \times 3p}{6p^2}$ **h.** $2x^{-2} \times 5xy$ **i.** $\dfrac{(3a)^{-2} \times 4a^2b^2}{2ab}$

3. Simplify.

a. $\lg 25 + \lg 4$ **b.** $\log_2 8 - \log_2 4$

c. $\log_5 50 + \log_5 2 - \log_5 4$ **d.** $\log_a 5 + \log_a 4 - \log_a 2$

e. $\log_x 4 + \log_x 12 - \log_x 3$ **f.** $\log_p a + \log_p b - \log_p c$

4. Solve the following equations.

a. $\log_5 x = \log_5 4 + \log_5 2$ **b.** $\log_5 4 - \log_5 2 = \log_5 x$

c. $\log_a 2 + \log_a x = \log_a 10$ **d.** $\log_3 x + \log_3 10 = \log_3 5 + \log_3 6 - \log_3 2$

e. $\lg 5 - \lg x + \lg 8 = \lg 4$ **f.** $\log_x 12 - \log_5 4 = \log_5 3$

2.1 Fractional Indices of a Power

Square root of 4 can be written either using the radical symbol (square root symbol) as $\sqrt{4}$ or using powers as $4^{\frac{1}{2}}$.

Therefore, it is clear that $\sqrt{4} = 4^{\frac{1}{2}}$.

Let us consider another example, similar to the above. As $2 = 2^1$,

$$\begin{aligned} 2 \times 2 \times 2 &= 2^1 \times 2^1 \times 2^1 \\ &= 2^3 \\ &= 8 \end{aligned}$$

Third power of 2 is 8. Thus, the cube root of 8 is 2. This can be denoted symbolically as,

$$\sqrt[3]{8} = 2 \text{ or } 8^{\frac{1}{3}} = 2.$$

Therefore it is clear that $\sqrt[3]{8} = 8^{\frac{1}{3}}$.

Futhermore, if a is a positive real number, then

$$\sqrt{a} = a^{\frac{1}{2}},$$
$$\sqrt[3]{a} = a^{\frac{1}{3}} \text{ and}$$
$$\sqrt[4]{a} = a^{\frac{1}{4}}.$$

Thus, the general relationship between the radical symbol and the exponent (index) of a power can be expressed as follows.

$$\boxed{\sqrt[n]{a} = a^{\frac{1}{n}}}$$

The following examples demonstrate how the above relationship can be used to simplify expressions involving powers.

Example 1

1. Find the value.

(i) $\sqrt[3]{27}$ (ii) $(\sqrt{25})^{-2}$ (iii) $\sqrt[3]{3\frac{3}{8}}$

(i) $\sqrt[3]{27} = 27^{\frac{1}{3}}$

$= (3^3)^{\frac{1}{3}}$

$= 3^{3 \times \frac{1}{3}}$

$= \underline{\underline{3}}$

(ii) $(\sqrt{25})^{-2} = (25^{\frac{1}{2}})^{-2}$

$= \{(5^2)^{\frac{1}{2}}\}^{-2}$

$= (5^{2 \times \frac{1}{2}})^{-2}$

$= 5^{-2}$

$= \frac{1}{5^2}$

$= \underline{\underline{\frac{1}{25}}}$

(iii) $\sqrt[3]{3\frac{3}{8}} = \sqrt[3]{\frac{27}{8}}$

$= \left(\frac{27}{8}\right)^{\frac{1}{3}}$

$= \frac{(3^3)^{\frac{1}{3}}}{(2^3)^{\frac{1}{3}}}$

$= \frac{3^{3 \times \frac{1}{3}}}{2^{3 \times \frac{1}{3}}}$

$= \frac{3}{2}$

$= \underline{\underline{1\frac{1}{2}}}$

The following examples further investigate how the laws of indices are used to simplify algebraic expressions involving powers.

Example 2

Simplify and express the answer with positive exponents (indices).

(i) $\left(\sqrt{x}\right)^3$ (ii) $\left(\sqrt[3]{a}\right)^{-\frac{1}{2}}$ (iii) $\frac{1}{\sqrt{x^{-3}}}$

(i) $\left(\sqrt{x}\right)^3 = \left(x^{\frac{1}{2}}\right)^3$

$= x^{\frac{1}{2} \times 3}$

$= \underline{\underline{x^{\frac{3}{2}}}}$

(ii) $\left(\sqrt[3]{a}\right)^{-\frac{1}{2}} = \left(a^{\frac{1}{3}}\right)^{-\frac{1}{2}}$

$= a^{\frac{1}{3} \times -\frac{1}{2}}$

$= a^{-\frac{1}{6}}$

$= \underline{\underline{\frac{1}{a^{\frac{1}{6}}}}}$

(iii) $\frac{1}{\sqrt{x^{-3}}} = \frac{1}{(x^{-3})^{\frac{1}{2}}}$

$= \frac{1}{x^{-3 \times \frac{1}{2}}}$

$= \frac{1}{x^{-\frac{3}{2}}}$

$= \underline{\underline{x^{\frac{3}{2}}}}$

Example 3

Find the value. (i) $\left(\frac{27}{64}\right)^{\frac{2}{3}}$ (ii) $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$

(i)
$$\begin{aligned}\left(\frac{27}{64}\right)^{\frac{2}{3}} &= \left(\frac{3^3}{4^3}\right)^{\frac{2}{3}}\\ &= \left[\left(\frac{3}{4}\right)^3\right]^{\frac{2}{3}}\\ &= \left(\frac{3}{4}\right)^{3\times\frac{2}{3}}\\ &= \left(\frac{3}{4}\right)^2\\ &= \underline{\underline{\frac{9}{16}}}\end{aligned}$$

(ii)
$$\begin{aligned}\left(\frac{16}{81}\right)^{-\frac{3}{4}} &= \left(\frac{2^4}{3^4}\right)^{-\frac{3}{4}}\\ &= \left(\frac{2}{3}\right)^{4\times-\frac{3}{4}}\\ &= \left(\frac{2}{3}\right)^{-3}\\ &= \left(\frac{3}{2}\right)^3\\ &= \frac{27}{8}\\ &= \underline{\underline{3\frac{3}{8}}}\end{aligned}$$

Let us now consider a slightly complex example: $\left(\frac{125}{64}\right)^{-\frac{1}{3}} \times \left(\sqrt[5]{32}\right)^3 \times 3^0$

$$\begin{aligned}\left(\frac{125}{64}\right)^{-\frac{1}{3}} \times \left(\sqrt[5]{32}\right)^3 \times 3^0 &= \left(\frac{5^3}{2^6}\right)^{-\frac{1}{3}} \times \left(32^{\frac{1}{5}}\right)^3 \times 1\\ &= \left(\frac{2^6}{5^3}\right)^{\frac{1}{3}} \times \left(2^{5\times\frac{1}{5}}\right)^3\\ &= \frac{2^{6\times\frac{1}{3}}}{5^{3\times\frac{1}{3}}} \times 2^3\\ &= \frac{2^2}{5} \times 2^3\\ &= \frac{2^5}{5}\\ &= \frac{32}{5}\\ &= \underline{\underline{6\frac{2}{5}}}\end{aligned}$$

Example 4

Simplify: $\dfrac{\sqrt[3]{343x^{3/2}}}{x}$

$$\begin{aligned}
\frac{\sqrt[3]{343x^{3/2}}}{x} &= \left(343x^{\frac{3}{2}}\right)^{\frac{1}{3}} \div x \\
&= 343^{\frac{1}{3}} \times \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} \div x \\
&= (7^3)^{\frac{1}{3}} \times \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} \div x \\
&= 7^1 \times x^{\frac{1}{2}} \div x \\
&= 7 \times x^{\frac{1}{2} - 1} \\
&= 7 \times x^{-\frac{1}{2}} \\
&= \underline{\underline{\frac{7}{x^{\frac{1}{2}}}}}
\end{aligned}$$

Exercise 2.1

1. Express the following using the radical symbol.

a. $p^{\frac{1}{3}}$ **b.** $a^{\frac{2}{3}}$ **c.** $x^{-\frac{2}{3}}$

d. $m^{\frac{4}{5}}$ **e.** $y^{-\frac{3}{4}}$ **f.** $x^{-\frac{5}{3}}$

2. Write using positive exponents (indices).

a. $\sqrt{m^{-1}}$ **b.** $\sqrt[3]{x^{-1}}$ **c.** $\sqrt[5]{p^{-2}}$ **d.** $(\sqrt{a})^{-3}$ **e.** $\sqrt[4]{x^{-3}}$

f. $(\sqrt[3]{p})^{-5}$ **g.** $\dfrac{1}{\sqrt{x^{-3}}}$ **h.** $\dfrac{1}{\sqrt[3]{a^{-2}}}$ **i.** $2\sqrt[3]{x^{-2}}$ **j.** $\dfrac{1}{3\sqrt{a^{-5}}}$

3. Find the value.

a. $\sqrt{25}$ **b.** $\sqrt[4]{16}$ **c.** $(\sqrt{4}\,)^5$

d. $(\sqrt[3]{27})^2$ **e.** $\sqrt[4]{81^3}$ **f.** $\sqrt[3]{1000^2}$

g. $\left(\frac{27}{125}\right)^{\frac{2}{3}}$ **h.** $\left(\frac{81}{10000}\right)^{\frac{3}{4}}$ **i.** $\left(\frac{1}{64}\right)^{-\frac{5}{6}}$

j. $\left(\frac{27}{64}\right)^{-\frac{2}{3}}$ **k.** $(0.81)^{\frac{3}{2}}$ **l.** $(0.125)^{-\frac{2}{3}}$

m. $\left(\frac{4}{25}\right)^{\frac{1}{2}} \times \left(\frac{3}{4}\right)^{-1} \times 2^0$ **n.** $\left(\frac{9}{100}\right)^{-\frac{3}{2}} \times \left(\frac{4}{25}\right)^{\frac{3}{2}}$ **o.** $(27)^{1\frac{1}{3}} \times (81)^{-1\frac{1}{4}}$

p. $\left(11\frac{1}{9}\right)^{-\frac{1}{2}} \times \left(6\frac{1}{4}\right)^{-\frac{3}{2}}$ **q.** $(0.125)^{-\frac{1}{3}} \times (0.25)^{\frac{3}{2}}$ **r.** $(\sqrt[3]{8}\,)^2 \times \sqrt[4]{16^3}$

4. Simplify and express using positive indices.

a. $\sqrt[3]{a^{-1}} \div \sqrt[3]{a}$ **b.** $\sqrt[5]{a^{-3}} \div \sqrt[5]{a^7}$ **c.** $\sqrt[3]{a^2} \div \sqrt[3]{a^3}$

d. $\left(\sqrt[3]{x^5}\right)^{\frac{1}{2}} \times \sqrt[6]{x^{-5}}$ **e.** $\left\{\left(\sqrt{a^3}\right)^{-2}\right\}^{\frac{-1}{2}}$ **f.** $\left(\sqrt{x^2y^2}\right)^{-6}$

g. $\sqrt{\frac{4a^{-2}}{9x^2}}$ **h.** $\left(\sqrt[3]{27x^3}\right)^{-2}$ **i.** $\left(\frac{xy^{-1}}{\sqrt{x^5}}\right)^{-2}$

2.2 Solving Equations with Indices

$2^x = 2^3$ is an equation. Because the bases of the powers on either side of the equal sign are equal, the exponents must be equal. Thus, from $2^x = 2^3$, we can conclude that $x = 3$.
Similarly, on either side of the equation $x^5 = 2^5$ are powers with equal exponents.

Because the indices are equal, the bases are also equal. Therefore, from $x^5 = 2^5$ we can conclude that $x = 2$. If $x^2 = 3^2$ then the indices are equal but in this case, both $+\,3$ and $-\,3$ are solutions. The reason for two solutions arising is because the exponent " 2 " is an even number. In this lesson, we will only consider powers with

a positive base. Thus, in expressions of the form x^m, $x > 0$.
There is a special property of powers of 1. All powers of 1 are equal to 1. That is, for any m, $1^m = 1$.

Let us summarise the above observations.

For $x > 0$, $y > 0$, $y \neq 1$ and $x \neq 1$,

$$\boxed{\begin{array}{l} \text{if } x \neq 0\ x^m = x^n, \text{ then } m = n. \\ \text{if } m \neq 0 \text{ and } x^m = y^m, \text{ then } x = y. \end{array}}$$

Let us use these rules to solve equations with indices.

Example 1

Solve.

(i) $4^x = 64$ (ii) $x^3 = 343$ (iii) $3 \times 9^{2x-1} = 27^{-x}$

(i) $4^x = 64$

$4^x = 4^3$

$\therefore \underline{\underline{x = 3}}$

(ii) $x^3 = 343$

$x^3 = 7^3$

$\therefore \underline{\underline{x = 7}}$

(iii)
$$\begin{aligned} 3 \times 9^{2x-1} &= 27^{-x} \\ 3 \times (3^2)^{2x-1} &= (3)^{3(-x)} \\ 3 \times 3^{2\,(2x-1)} &= 3^{-3x} \\ 3^{\,1+4x-2} &= 3^{-3x} \\ \therefore\ 1 + 4x - 2 &= -3x \\ 4x + 3x &= 2 - 1 \\ 7x &= 1 \\ x &= \underline{\underline{\frac{1}{7}}} \end{aligned}$$

Exercise 2.2

1. Solve each of the following equations.

a. $3^x = 9$ **b.** $3^{x+2} = 243$

c. $4^{3x} = 32$ **d.** $2^{5x-2} = 8^x$

e. $8^{x-1} = 4^x$ **f.** $x^3 = 216$

g. $2\sqrt{x} = 6$ **h.** $\sqrt[3]{2x^2} = 2$

2. Solve each of the following equations.

a. $2^x \times 8^x = 256$ **b.** $8 \times 2^{x-1} = 4^{x-2}$

c. $5 \times 25^{2x-1} = 125$ **d.** $3^{2x} \times 9^{3x-2} = 27^{-3x}$

e. $4^x = \frac{1}{64}$ **f.** $(3^x)^{-\frac{1}{2}} = \frac{1}{27}$

g. $3^{4x} \times \frac{1}{9} = 9^x$ **h.** $x^2 = \left(\frac{1}{8}\right)^{-\frac{2}{3}}$

2.3 Laws of logarithms

We know that, using the laws of logarithms, we can write

$\log_2 (16 \times 32) = \log_2 16 + \log_2 32$ and $\log_2 (32 \div 16) = \log_2 32 - \log_2 16$. These laws, in general, can be written as follows.

$$\log_a (mn) = \log_a m + \log_a n$$

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Let us learn another law of a similar type.

Consider $\log_5 125^4$ as an example.

$$\begin{aligned}\log_5 125^4 &= \log_5 (125 \times 125 \times 125 \times 125)\\ &= \log_5 125 + \log_5 125 + \log_5 125 + \log_5 125\\ &= 4 \log_5 125\end{aligned}$$

Similarly,

$$\log_{10} 10^5 = 5 \log_{10} 10 \text{ and}$$

$$\log_3 5^2 = 2 \log_3 5$$

This observation can be written, in general, as the following logarithmic law.

$$\boxed{\log_a m^r = r \log_a m}$$

This law is even valid for expressions with fractional indices. Given below are a few examples, where this law is applied to powers with fractional indices.

$$\log_2 3^{\frac{1}{2}} = \frac{1}{2} \log_2 3$$

$$\log_5 7^{\frac{2}{3}} = \frac{2}{3} \log_5 7$$

The following examples consider how all the laws of logarithms that you have learned so far, including the above, are used.

Example 1

Evaluate.

(i) $\lg 1000$ (ii) $\log_4 \sqrt[3]{64}$ (iii) $2 \log_2 2 + 3 \log_2 4 - 2 \log_2 8$

(i) $$\begin{aligned}\lg 1000 &= \lg 10^3\\ &= 3 \lg 10\\ &= 3 \times 1 \quad (\text{because } \lg 10 = 1)\\ &= \underline{\underline{3}}\end{aligned}$$

(ii) $\log_4 \sqrt[3]{64} = \log_4 64^{\frac{1}{3}}$

$= \frac{1}{3} \log_4 64$

$= \frac{1}{3} \log_4 4^3$

$= \frac{1}{3} \times 3 \log_4 4$

$= \log_4 4$

$= \underline{\underline{1}}$

(iii) $2 \log_2 2 + 3 \log_2 4 - 2\log_2 8 = 2\log_2 2 + 3 \log_2 2^2 - 2\log_2 2^3$

$= \log_2 2^2 + \log_2 (2^2)^3 - \log_2 (2^3)^2$

$= \log_2 \left(\frac{2^2 \times (2^2)^3}{(2^3)^2}\right)$

$= \log_2 \left(\frac{2^2 \times 2^6}{2^6}\right)$

$= \log_2 2^2$

$= 2 \log_2 2$

$= \underline{\underline{2}}$

Example 2

Solve.

(i) $2\lg 8 + 2\lg 5 = \lg 4^3 + \lg x$

$\therefore \quad \lg x = 2\lg 8 + 2\lg 5 - \lg 4^3$

$= \lg 8^2 + \lg 5^2 - \lg 4^3$

$\therefore \quad \lg x = \lg \left(\frac{8^2 \times 5^2}{4^3}\right)$

$\therefore \quad \lg x = \lg 25$

$\therefore \quad \underline{\underline{x = 25}}$

(ii) $2 \log_b 3 + 3 \log_b 2 - \log_b 72 = \frac{1}{2} \log_b x$

$\therefore \; 2 \log_b 3 + 3 \log_b 2 - \log_b 72 = \frac{1}{2} \log_b x$

$\therefore \; \log_b 3^2 + \log_b 2^3 - \log_b 72 = \log_b x^{\frac{1}{2}}$

$\therefore \; \log_b \left(\frac{3^2 \times 2^3}{72}\right) = \log_b x^{\frac{1}{2}}$

$\therefore \; \frac{3^2 \times 2^3}{72} = x^{\frac{1}{2}}$

$\therefore \; 1^2 = (x^{\frac{1}{2}})^2$

$\therefore \; 1 = x^1$

$\therefore \; \underline{\underline{x = 1}}$

Example 3

Verify: $\log_5 75 - \log_5 3 = \log_5 40 - \log_5 8 + 1$

$$\begin{aligned}
\text{Left Side} &= \log_5 \frac{75}{3} \\
&= \log_5 25 \\
&= \log_5 5^2 \\
&= 2 \\
\text{Right Side} &= \log_5 40 - \log_5 8 + 1 \\
&= \log_5 \frac{40}{8} + 1 \\
&= \log_5 5 + 1 \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

$\therefore \; \log_5 75 - \log_5 3 = \log_5 40 - \log_5 8 + 1$

Use the laws of logarithms to do the following exercise.

Exercise 2.3

1. Evaluate.

a. $\log_2 32$ **b.** $\lg 10000$ **c.** $\frac{1}{3} \log_3 27$

d. $\frac{1}{2} \log_5 \sqrt{25}$ **e.** $\log_3 \sqrt[4]{81}$ **f.** $3 \log_2 \sqrt[3]{8}$

2. Simplify each of the following expressions and find the value.

a. $2 \log_2 16 - \log_2 8$

b. $\lg 80 - 3 \lg 2$

c. $2 \lg 5 + 3 \lg 2 - \lg 2$

d. $\lg 75 - \lg 3 + \lg 28 - \lg 7$

e. $\lg 18 - 3 \lg 3 + \frac{1}{2} \lg 9 + \lg 5$

f. $4 \lg 2 + \lg \frac{15}{4} - \lg 6$

g. $\lg \frac{1}{256} - \lg \frac{125}{4} - 3 \lg \frac{1}{20}$

h. $\log_3 27 + 2 \log_3 3 - \log_3 3$

i. $\lg \frac{12}{5} + \lg \frac{25}{21} - \lg \frac{2}{7}$

j. $\lg \frac{3}{4} - 2 \lg \frac{3}{10} + \lg 12 - 2$

3. Solve the following equations.

a. $\log x + \lg 4 = \lg 8 + \lg 2$

b. $4 \lg 2 + 2 \lg x + \lg 5 = \lg 15 + \lg 12$

c. $3 \lg x + \log 96 = 2 \lg 9 + \lg 4$

d. $\lg x = \frac{1}{2} (\lg 25 + \lg 8 - \lg 2)$

e. $3 \lg x + 2 \lg 8 = \lg 48 + \frac{1}{2} \lg 25 - \lg 30$

f. $\lg 125 + 2 \lg 3 = 2 \lg x + \lg 5$

Summary

- $\sqrt[n]{a} = a^{\frac{1}{n}}$

- If $x > 0, y > 0$ and $x \neq 1, y \neq 1$

  $x \neq 0$ and $x^m = x^n$, then $m = n$.
  $m \neq 0$ and $x^m = y^m$, then $x = y$.

- $\log_a m^r = r \log_a m$

Miscellaneous Exercise

1. Find the value.

a. $(\sqrt[3]{8})^2 \times \dfrac{1}{\sqrt[3]{27}}$

b. $(\sqrt{125})^3 \times \dfrac{1}{\sqrt{20}} \times 10$

c. $\dfrac{32^{-\frac{2}{5}} \times 216^{\frac{2}{3}}}{81^{\frac{3}{4}} \times \sqrt[3]{8^0} \times \sqrt[3]{27^{-2}}}$

d. $\sqrt{\dfrac{18 \times 5^2}{8}}$

e. $\left(\dfrac{1}{8}\right)^{-\frac{1}{3}} \times 5^{-2} \times 100$

f. $27^{\frac{2}{3}} - 16^{\frac{3}{4}}$

2. Simplify and express using positive indices.

a. $\sqrt{a^2 b^{-\frac{1}{2}}}$

b. $(x^{-4})^{\frac{1}{2}} \times \dfrac{1}{\sqrt{x^{-3}}}$

c. $(x^{\frac{1}{2}} - x^{-\frac{1}{2}})\,(x^{\frac{1}{2}} + x^{-\frac{1}{2}})$

d. $(x \div \sqrt[n]{x}\,)^n$

e. $\left[\left(\sqrt{a^3}\right)^{-2}\right]^{\frac{1}{2}}$

3. Verify the following.

a. $\lg\left(\dfrac{217}{38} \div \dfrac{31}{266}\right) = 2\lg 7$

b. $\dfrac{1}{2}\lg 9 + \lg 2 = 2\lg 3 - \lg 1.5$

c. $\log_3 24 + \log_3 5 - \log_3 40 = 1$

d. $\lg 26 + \lg 119 - \lg 51 - \lg 91 = \lg 2 - \lg 3$

e. $2\log_a 3 + \log_a 20 - \log_a 36 = \log_a 10 - \log_a 2$

3 Indices and Logarithms II

By studying this lesson you will be able to

- use the table of logarithms to simplify expressions involving products and quotients of powers and roots of numbers between 0 and 1.
- identify the two keys $\boxed{\wedge}$ and $\boxed{\sqrt{\ }}$ on a scientific calculater, and simplify expressions involving decimal numbers, powers and roots using a scientific calculator.

Logarithms

$10^3 = 1000$ can be written using logarithms as $\log_{10} 1000 = 3$. As a convention we write "lg" instead of "$\log_{10}$". Now we can express the above expression as $\lg 1000 = 3$. It is important to mention the base if it is other than 10.

For example,

$\log_5 25 = 2$ because $5^2 = 25$,
$\lg 1 = 0$ because $10^0 = 1$, and
$\lg 10 = 1$ because $10^1 = 10$.

The logarithm of any positive number can be found using the table of logarithms. Do the following exercise to refresh the memory on using logarithms to simplify expressions involving multiplications and divisions of numbers.

Review Exercise

1. Complete the following tables.

(i)

Number	Scientific notation	Logarithm		Logarithm
		Characteristic	Mantissa	
73.45	7.345×10^1	1	0.8660	1.8660
8.7				
12.5				
725.3				
975				

(ii)

Logarithm	Logarithm		Scientific notation	Number
	Characteristic	Mantissa		
1.5492				
2.9059				
1.4036				
2.8798				
3.4909				

2. Use the table of logarithms to fill in the blanks.

a. lg 5.745 = 0.7593, therefore 5.745 = $10^{0.7593}$
b. lg 9.005 = ..., therefore 9.005 = $10^{\cdots\cdots}$
c. lg 82.8 = ..., therefore 82.8 = $10^{\cdots\cdots}$
d. lg 74.01 = ..., therefore 74.01 = $10^{\cdots\cdots}$
e. lg 853.1 = ..., therefore 853.1 = $10^{\cdots\cdots}$
f. antilog 0.7453 = 5.562, therefore 5.562 = $10^{0.7453}$
g. antilog 0.0014 = ..., therefore .... = $10^{0.0014}$
h. antilog 1.9251 = ..., therefore .... = $10^{1.9251}$
i. antilog 2.4374 = ..., therefore .... = $10^{2.4374}$
j. antilog 3.2001 = ..., therefore .... = $10^{3.2001}$

3. Fill in the blanks and find the value of P.

(i) In terms of logarithms

$$P = \frac{27.32 \times 9.8}{11.5}$$

lg P = lg ...... + lg ...... − lg ....

= .... + ... − ....

= ....

$\therefore$ P = antilog ....

= ....

(ii) Using indices

$$P = \frac{27.32 \times 9.8}{11.5}$$

$$= \frac{10^{\cdots\cdots} \times 10^{\cdots\cdots}}{10^{\cdots\cdots}}$$

$$= \frac{10^{\cdots\cdots}}{10^{\cdots\cdots}}$$

$= 10^{\cdots\cdots}$

= .... $\times 10^{\cdots\cdots}$

= ....

4. Simplify the expressions using logarithms.

a. 14.3×95.2 **b.** $2.575 \times 9.27 \times 12.54$ **c.** $\dfrac{9.87 \times 7.85}{4.321}$

3.1 Logarithms of decimal numbers less than one

Let us now consider how to use the table of logarithms to obtain the logarithms of numbers between 0 and 1, by paying close attention to how we obtained the logarithms of numbers greater than 1. For this purpose, carefully investigate the following table.

Number	Scientific Notation	Logarithm		Logarithm
		Characteristic	Mantissa	
5432	5.432×10^3	3	0.7350	3.7350
543.2	5.432×10^2	2	0.7350	2.7350
54.32	5.432×10^1	1	0.7350	1.7350
5.432	5.432×10^0	0	0.7350	0.7350
0.5432	5.432×10^{-1}	-1	0.7350	$\bar{1}.7350$
0.05432	5.432×10^{-2}	-2	0.7350	$\bar{2}.7350$
0.005432	5.432×10^{-3}	-3	0.7350	$\bar{3}.7350$
0.0005432	5.432×10^{-4}	-4	0.7350	$\bar{4}.7350$

According to the above table, the characteristic of the logarithm of numbers inbetween 0 and 1, coming after 5.432 in the first column, are negative. Even though the characteristic is negative, the mantissa of the logarithm, which is found using the table, is a positive number. The symbol "–" is used above the whole part to indicate that only the characteristic is negative. It is read as "**bar**".

For example, $\bar{2}.3725$ is read as "bar two point three, seven, two, five". Moreover, what is represented by $\bar{2}.3725$ is $-2 + 0.3725$.

The characteristic of the logarithm of a number between 0 and 1 is negative. The characteristic of the logarithm of such a number can be obtained either by writing it in scientific notation or by counting the number of zeros after the decimal point.

The characteristic of the logarithm can be obtained by adding one to the number of zeros after the decimal point (and before the next non-zero digit) and taking its negative value. Observe it in the above table too.

Example

0.004302 Number of zeros after the decimal point and before the next non-zero digit is 2. Therefore characteristic of the logarithm is $\bar{3}$

0.04302	Number of zeros after the decimal point is 1; thus the characteristic of the logarithm is $\overline{2}$
0.4302	Number of zeros after the decimal point is 0; thus the characteristic of the logarithm is $\overline{1}$

Therefore, $\lg 0.004302 = \overline{3}.6337$.

When written using indices, it is;
$0.004302 = 10^{\overline{3}.6337}$. This can also be written as, $0.004302 = 10^{-3} \times 10^{0.6337}$.

Do the following exercise to practice taking logarithms of numbers between 0 and 1.

Exercise 3.1

1. For each of the following numbers, write the characteristic of its logarithm.

a. 0.9843 **b.** 0.05 **c.** 0.0725
d. 0.0019 **e.** 0.003141 **f.** 0.000783

2. Find the value.

a. lg 0.831 **b.** lg 0.01175 **c.** lg 0.0034
d. lg 0.009 **e.** lg 0.00005 **f.** lg 0.00098

3. Express each of the following numbers as a power of 10.

a. 0.831 **b.** 0.01175 **c.** 0.0034
d. 0.009 **e.** 0.00005 **f.** 0.00098

3.2 Number corresponding to a logarithm (antilog)

Let us recall how the antilog of a number greater than 1 is obtained.

$$\begin{aligned} \text{antilog } 2.7421 &= 5.522 \times 10^2 \\ &= 552.2 \end{aligned}$$

When a number is written in scientific form, the index of the power of 10 is the characteristic of the logarithm of that number. The characteristic of the logarithm indicates the number of places that the decimal point needs to be shifted when taking the antilog.

Thus, we obtained 552.2 by shifting the decimal point of 5.522 two places to the right. However, when the characteristic is negative the decimal point is shifted to the left side.

$\text{antilog } \bar{2}.7421 = 5.522 \times 10^{-2}$ (Decimal point needs to be shifted 2 places to the left)

$= 0.05522$ (Because of bar 2, there is one 0 after the decimal point)

$\text{antilog } \bar{1}.7421 = 5.522 \times 10^{-1}$ (Decimal place needs to be shifted one place to the left)

$= 0.5522$ (Because of bar 1, there are no zeros after the decimal place)

Exercise 3.2

1. Express each of the following numbers, given in the scientific form, in decimal form.

a. 3.37×10^{-1} **b.** 5.99×10^{-3} **c.** 6.0×10^{-2}

d. 5.745×10^{0} **e.** 9.993×10^{-4} **f.** 8.777×10^{-3}

2. Find the value using the logarithmic table.

a. $\text{antilog } \bar{2}.5432$ **b.** $\text{antilog } \bar{1}.9321$ **c.** $\text{antilog } 0.9972$

d. $\text{antilog } \bar{4}.5330$ **e.** $\text{antilog } \bar{2}.0000$ **f.** $\text{antilog } \bar{3}.5555$

3.3 Addition and subtraction of logarithms with negative characteristics

(a) Addition

The mantissa of a logarithm is obtained from the table of logarithms and is always positive. But we now know that, the characteristic can be positive, negative or zero. In $\bar{2}.5143$, the mantissa, .5143, is positive and the characteristic, $\bar{2}$, is negative. When adding or subtracting such numbers, it is important to simplify the characteristic and the mantissa separately.

Example 1

Simplify and express the answer in log form.

(i) $\bar{2}.5143 + \bar{1}.2375$ (ii) $\bar{3}.9211 + 2.3142$ (iii) $\bar{3}.8753 + \bar{1}.3475$

(i) $\bar{2}.5143 + \bar{1}.2375$

$$= -2 + 0.5143 + (-1) + 0.2375$$
$$= (-2 - 1) + (0.5143 + 0.2375)$$
$$= -3 + 0.7518$$
$$= \underline{\underline{\bar{3}.7518}}$$

(ii) $\bar{3}.9211 + 2.3142$ $= -3 + 0.9211 + 2 + 0.3142$
$= (-3 + 2) + (0.9211 + 0.3142)$
$= -1 + 1.2353$
$= -1 + 1 + 0.2353$
$= \underline{\underline{0.2353}}$

(iii) $\bar{3}.8753 + 1.3475$ $= -3 + 0.8753 + 1 + 0.3475$
$= (-3 + 1) + (0.8753 + 0.3475)$
$= -2 + 1.2228$
$= -2 + 1 + 0.2228$
$= \underline{\underline{\bar{1}.2228}}$

(b) Subtraction

As in addition, logarithms should be subtracted from right to left, remembering that the mantissa is positive.

Example 2

Simplify and express the answer in log form.

(i) $\bar{2}.5143 - 1.3143 = -2 + 0.5143 - (1 + 0.3143)$
$= -2 + 0.5143 - 1 - 0.3143$
$= -2 - 1 + 0.5143 - 0.3143$
$= -3 + 0.2000$
$= \underline{\underline{\bar{3}.2000}}$

(ii) $2.5143 - \bar{1}.9143 = 2 + 0.5143 - (-1 + 0.9143)$
$= 2 + 0.5143 + 1 - 0.9143$
$= 3 - 0.4000$
$= \underline{\underline{2.6000}}$

(iii) $0.2143 - \bar{1}.8143 = 0.2143 - (-1 + 0.8143)$
$= 0.2143 + 1 - 0.8143$
$= 1 - 0.6000$
$= \underline{\underline{0.4}}$

(iv) $\bar{2}.5143 - \bar{1}.9143 = -2 + 0.5143 - (-1 + 0.9143)$
$= -2 + 0.5143 + 1 - 0.9143$
$= -2 + 1 + 0.5143 - 0.9143$
$= -1 - 0.4000$

In the above example, the decimal part is negative. Because we need the decimal part of a logarithm to be positive, we will use a trick as follows to make it positive.

$$-1 - 0.4 = -1 - 1 + 1 - 0.4 \quad \text{(Because } -1 + 1 = 0\text{, we have not changed the value.)}$$
$$= -2 + 0.6$$
$$= \bar{2}.6$$

Actually what we have done is add − 1 to the characteristic and + 1 to the mantissa.

Note: We could have avoided getting a negative decimal part by doing the simplification as follows.

$$-2 + 0.5143 + 1 - 0.9143 = -2 + 1.5143 - 0.9143 = -2 + 0.6 = \bar{2}.6$$

Exercise 3.3

1. Simplify

a. $0.7512 + \bar{1}.3142$ **b.** $\bar{1}.3072 + \bar{2}.2111$ **c.** $\bar{2}.5432 + \bar{1}.9513$

d $\bar{3}.9121 + \bar{1}.5431$ **e.** $0.7532 + \bar{3}.8542$ **f.** $\bar{1}.8311 + \bar{2}.5431 + 1.3954$

g. $3.8760 - \bar{2}.5431$ **h.** $\bar{2}.5132 - \bar{1}.9332$ **i.** $\bar{3}.5114 - \bar{2}.4312$

j $\bar{2}.9372 - 1.5449$ **k.** $0.7512 + \bar{1}.9431$ **l.** $\bar{1}.9112 - \bar{3}.9543$

2. Simplify and express in log form.

a. $\bar{1}.2513 + 0.9172 - \bar{1}.514$ **b.** $\bar{3}.2112 + 2.5994 - \bar{1}.5004$

c. $\bar{3}.2754 + \bar{2}.8211 - \bar{1}.4372$ **d.** $0.8514 - \bar{1}.9111 - \bar{2}.3112$

e. $\bar{3}.7512 - (0.2511 + \bar{1}.8112)$ **f.** $\bar{1}.2572 + 3.9140 - \bar{1}.1111$

3.4 Simplification of numerical expressions using the table of logarithms

The following examples show how numerical computations are done using the given logarithmic rules.

1. $\log_a (P \times Q) = \log_a P + \log_a Q$

2. $\log_a \left(\frac{P}{Q}\right) = \log_a P - \log_a Q$

Example 1

Simplify using the table of logarithms and logarithmic rules.

a. 43.85×0.7532 **b.** 0.0034×0.8752

c. $0.0875 \div 18.75$ **d.** $0.3752 \div 0.9321$

Two methods of simplifying are shown below.

Method 1:

a. 43.85×0.7532

Take $P = 43.85 \times 0.7532$

$$\begin{aligned}
\text{Then } \lg P &= \lg (43.85 \times 0.7532)\\
&= \lg 43.85 + \lg 0.7532\\
&= 1.6420 + \bar{1}.8769\\
&= 1 + 0.6420 - 1 + 0.8769\\
&= 1.5189\\
\therefore \quad P &= \text{antilog } 1.5189\\
&= \underline{\underline{33.03}}
\end{aligned}$$

Method 2:

Simplifying using indices

$$\begin{aligned}
& 43.85 \times 0.7532\\
&= 10^{1.6420} \times 10^{\bar{1}.8769}\\
&= 10^{1.5189}\\
&= 3.303 \times 10^{1}\\
&= \underline{\underline{33.03}}
\end{aligned}$$

b. 0.0034×0.8752

Take $P = 0.0034 \times 0.8752$.

Then,

$$\begin{aligned}
\lg P &= \lg (0.0034 \times 0.8752)\\
&= \lg 0.0034 + \lg 0.8752\\
&= \bar{3}.5315 + \bar{1}.9421\\
&= -3 + 0.5315 - 1 + 0.9421\\
&= -4 + 1.4736\\
&= -4 + 1 + 0.4736\\
&= -3 + 0.4736\\
&= \bar{3}.4736\\
\therefore \quad P &= \text{antilog } \bar{3}.4736\\
&= \underline{\underline{0.002975}}
\end{aligned}$$

Simplifying using indices

$$\begin{aligned}
& 0.0034 \times 0.8752\\
&= 10^{\bar{3}.5315} \times 10^{\bar{1}.9421}\\
&= 10^{\bar{3}.4736}\\
&= 2.975 \times 10^{-3}\\
&= \underline{\underline{0.002975}}
\end{aligned}$$

c. $0.0875 \div 18.75$

Take $P = 0.0875 \div 18.75$

Then, $\lg P = \lg (0.0875 \div 18.75)$

$= \lg 0.0875 - \lg 18.75$

$= \overline{2}.9420 - 1.2730$

$= -2 + 0.9420 - 1 - 0.2730$

$= -3 + 0.6690$

$= \overline{3}.6690$

$\therefore \quad P = \text{antilog } \overline{3}.6690$

$= \underline{\underline{0.004666}}$

Simplifying using indices
$0.0875 \div 18.75$
$= 10^{\overline{2}.9420} \div 10^{1.2730}$
$= 10^{\overline{2}.9420 - 1.2730}$
$= 10^{\overline{3}.6690}$
$= 4.666 \times 10^{-3}$
$= \underline{\underline{0.004666}}$

d. $0.3752 \div 0.9321$

Take $P = 0.3752 \div 0.9321$

Then, $\lg P = \lg (0.3752 \div 0.9321)$

$= \lg 0.3752 - \lg 0.9321$

$= \overline{1}.5742 - \overline{1}.9694$

$= -1 + 0.5742 - (-1 + 0.9694)$

$= -1 + 0.5742 + 1 - 0.9694$

$= -1 + 0.5742 + 0.0306$

$= -1 + 0.6048$

$= \overline{1}.6048$

$\therefore \quad P = \text{antilog } \overline{1}.6048$

$= \underline{\underline{0.4026}}$

Simplifying using indices
$0.3752 \div 0.9321$
$= 10^{\overline{1}.5742} \div 10^{\overline{1}.9694}$
$= 10^{\overline{1}.5742 - \overline{1}.9694}$
$= 10^{\overline{1}.6048}$
$= 4.026 \times 10^{-1}$
$= \underline{\underline{0.4026}}$

Example 2

Simplify using the table of logarithms.

$$\frac{8.753 \times 0.02203}{0.9321}$$

Take $P = \dfrac{8.753 \times 0.02203}{0.9321}$.

Then, $\lg P = \lg \left(\dfrac{8.753 \times 0.02203}{0.9321}\right)$

$= \lg 8.753 + \lg 0.02203 - \lg 0.9321$

$= 0.9421 + \bar{2}.3430 - \bar{1}.9694$

$= 0.9421 - 2 + 0.3430 - \bar{1}.9694$

$= \bar{1}.2851 - \bar{1}.9694$

$= -1 + 0.2851 - (-1 + 0.9694)$

$= -1 + 0.2851 + 1 - 0.9694$

$= \bar{1}.3157$

$\therefore \ P = \text{antilog } \bar{1}.3157$

$= \underline{\underline{0.2068}}$

Simplifying using indices

$$\frac{8.753 \times 0.02203}{0.9321}$$

$$= \frac{10^{0.9421} \times 10^{\bar{2}.3430}}{10^{\bar{1}.9694}}$$

$$= \frac{10^{\bar{1}.2851}}{10^{\bar{1}.9694}}$$

$$= 10^{\bar{1}.2851 - \bar{1}.9694}$$

$$= 10^{\bar{1}.3157}$$

$$= 2.067 \times 10^{-1}$$

$$= \underline{\underline{0.2068}}$$

Exercise 3.4

Find the value using the table of logarithms.

1. **a.** 5.945×0.782 **b.** 0.7453×0.05921 **c.** 0.0085×0.0943
d. $5.21 \times 0.752 \times 0.093$ **e.** $857 \times 0.008321 \times 0.457$ **f.** $0.123 \times 0.9857 \times 0.79$

2.
a. $7.543 \div 0.9524$ **b.** $0.0752 \div 0.8143$ **c.** $0.005273 \div 0.0078$
d. $0.9347 \div 8.75$ **e.** $0.0631 \div 0.003921$ **f.** $0.0752 \div 0.0008531$

3.
a. $\dfrac{8.247 \times 0.1973}{0.9875}$ **b.** $\dfrac{9.752 \times 0.0054}{0.09534}$ **c.** $\dfrac{79.25 \times 0.0043}{0.3725}$

d. $\dfrac{0.7135 \times 0.4391}{0.0059}$ **e.** $\dfrac{5.378 \times 0.9376}{0.0731 \times 0.471}$ **f.** $\dfrac{71.8 \times 0.7823}{23.19 \times 0.0932}$

3.5 Multiplication and division of a logarithm of a number by a whole number

We know that the characteristic of a number greater than one is positive. Multiplying or dividing such a logarithm by a number can be done in the usual way.

We know that the characteristic of the logarithm of a number between 0 and 1 is negative. $\overline{3}.8247$ is such a logarithm. When multiplying or dividing a logarithm with a negative characteristic by a number, we simplify the characteristic and the mantissa separately.

Multiplication of a logarithm by a whole number

Example 1

Simplify.

a. 2.8111×2 **b.** $\overline{2}.7512 \times 3$ **c.** $\overline{1}.9217 \times 3$

a.

$$\begin{aligned} & 2.8111 \times 2 \\ = \; & \underline{\underline{5.6222}} \end{aligned}$$

b.

$$\begin{aligned} & \overline{2}.7512 \times 3 \\ &= 3(-2 + 0.7512) \\ &= -6 + 2.2536 \\ &= -6 + 2 + 0.2536 \\ &= -4 + 0.2536 \\ &= \underline{\underline{\overline{4}.2536}} \end{aligned}$$

c.

$$\begin{aligned} & \overline{1}.9217 \times 3 \\ & 3(-1 + 0.9217) \\ &= -3 + 2.7651 \\ &= -3 + 2 + 0.7651 \\ &= -1 + 0.7651 \\ &= \underline{\underline{\overline{1}.7651}} \end{aligned}$$

Division of a logarithm by a whole number

Let us now consider how to divide a logarithm by a whole number. When the characteristic of a logarithm is negative the characteristic and the mantissa carry negative and positive values respectively. Therefore, it is important to divide the positive part and the negative part separately. Let us now consider some examples of this type.

Example 2

Simplify.

a. $2.5142 \div 2$

b. $\bar{3}.5001 \div 3$

because, $(-3 + 0.5001) \div 3$

c. $\bar{4}.8322 \div 2$

because, $(-4 + 0.8322) \div 2$

$2.5142 \div 2$

$= \underline{\underline{1.2571}}$

$\bar{3} \div 3 = \bar{1}$

$0.5001 \div 3 = 0.1667$

$\therefore \bar{3}.5001 \div 3$

$= \underline{\underline{\bar{1}.1667}}$

$\bar{4} \div 2 = \bar{2}$

$0.8322 \div 2 = 0.4161$

$\therefore \bar{4}.8322 \div 2$

$= \underline{\underline{\bar{2}.4161}}$

In the above example, the characteristic of the logarithm was perfectly divisible. Let us consider in the following example, how division is done when the whole part is not perfectly divisible.

Example 3

Simplify.

a. $\bar{1}.5412 \div 2$ **b.** $\bar{1}.3712 \div 3$ **c.** $\bar{3}.5112 \div 2$

a. $\bar{1}.5412 \div 2$ can be written as $(-1 + 0.5412) \div 2$.

Because the whole part, $\bar{1}$, is not perfectly divisible by 2, let us write it as $\bar{2} + 1$. Now, we can perform the division as follows

a. $\bar{1}.5412 \div 2 = (-1 + 0.5412) \div 2$

$= (-2 + 1 + 0.5412) \div 2$

$= (-2 + 1.5412) \div 2$

$= \underline{\underline{\bar{1}.7706}}$

b. $\bar{1}.3712 \div 3$

$= (-1 + 0.3712) \div 3$

$= (-3 + 2 + 0.3712) \div 3$ beacause $(-1 = -3 + 2)$

$= (\bar{3} + 2.3712) \div 3$

$= \underline{\underline{\bar{1}.7904}}$

c. $\bar{3}.5112 \div 2$

$= (-3 + 0.5112) \div 2$

$= (-4 + 1 + 0.5112) \div 2$ because $(-3 = -4 + 1)$

$= \bar{4} + 1.5112 \div 2$

$= \underline{\underline{\bar{2}.7556}}$

These types of divisions and multiplications are important when simplifying using the table of logarithms. Do the following exercise to strengthen this knowledge.

Exercise 3.5

1. Find the value.

a. $\overline{1}.5413 \times 2$ **b.** $\overline{2}.7321 \times 3$ **c.** 1.7315×3

d. 0.4882×3 **e.** $\overline{3}.5111 \times 2$ **f.** $\overline{3}.8111 \times 4$

2. Find the value.

a. $1.9412 \div 2$ **b.** $0.5512 \div 2$ **c.** $\overline{2}.4312 \div 2$

d. $\overline{3}.5412 \div 3$ **e.** $\overline{2}.4712 \div 2$ **f.** $\overline{4}.5321 \div 2$

g. $\overline{1}.5432 \div 2$ **h.** $\overline{2}.9312 \div 3$ **i.** $\overline{3}.4112 \div 2$

j. $\overline{1}.7512 \div 3$ **k.** $\overline{4}.1012 \div 3$ **l.** $\overline{5}.1421 \div 3$

3.6 Finding powers and roots of numbers using the table of logarithms

Recall that $\log_2 5^3 = 3 \log_2 5$.

This follows from the logarithmic rule $\log_a m^r = r \log_a m$.

Similarly, the logarithm of a root can be written using this rule, as follows.

(i) $\log_a \sqrt{5} = \log_a 5^{\frac{1}{2}}$ (because $\sqrt{5} = 5^{\frac{1}{2}}$)

$= \underline{\underline{\frac{1}{2} \log_a 5}}$ (using the above logarithmic rule)

(ii) $\lg \sqrt{25} = \lg 25^{\frac{1}{2}}$

$= \underline{\underline{\frac{1}{2} \lg 25}}$

The following examples consider how to extract roots and powers of a number using the table of logarithms.

Example 1

Find the value.

a. 354^2 **b.** 0.0275^3 **c.** 0.9073^4

a. **Take** $P = 354^2$.

$\lg P = \lg 354^2$

$= 2 \lg 354$

$= 2 \lg 3.54 \times 10^2$

$= 2 \times 2.5490$

$= 5.0980$

$\therefore \quad P = \text{antilog } 5.0980$

$= 1.253 \times 10^5$

$= \underline{\underline{125\,300}}$

b. **Take** $P = 0.0275^3$.

$\lg P = \lg 0.0275^3$

$= 3 \lg 0.0275$

$= 3 \times \overline{2}.4393$

$= 3 \times (-2 + 0.4393)$

$= -6 + 1.3179$

$= -6 + 1 + 0.3179$

$= -5 + 0.3179$

$= \overline{5}.3179$

$\therefore \quad P = \text{antilog } \overline{5}.3179$

$= 2.079 \times 10^{-5}$

$= \underline{\underline{0.00002079}}$

c. **Take** $P = 0.9073^4$.

$\lg P = \lg 0.9073^4$

$= 4 \lg 0.9073$

$= 4 \times \overline{1}.9577$

$= 4 \times (-1 + 0.9577)$

$= -4 + 3.8308$

$= -4 + 3 + 0.8308$

$= -1 + 0.8308$

$= \overline{1}.8308$

$\therefore P = \text{antilog } \overline{1}.8308$

$= 6.773 \times 10^{-1}$

$= \underline{\underline{0.6773}}$

Simplifying using indices

$0.9073^4 = \left(10^{\overline{1}.9577}\right)^4$

$= 10^{\overline{1}.9577 \times 4}$

$= 10^{\overline{1}.8308}$

$= 6.773 \times 10^{-1}$

$= \underline{\underline{0.6773}}$

Example 2

(i) $\sqrt{8.75}$ (ii) $\sqrt[3]{0.9371}$ (iii) $\sqrt[3]{0.0549}$

(i) Take $P = \sqrt{8.75}$.

$P = \sqrt{8.75}$ can be written as

$$P = 8.75^{\frac{1}{2}}$$

$$\lg P = \lg 8.75^{\frac{1}{2}}$$

$$= \frac{1}{2} \lg 8.75$$

$$= \frac{1}{2} \times 0.9420$$

$$= 0.4710$$

$$\therefore \quad P = \text{antilog } 0.4710$$

$$= \underline{\underline{2.958}}$$

(ii) Take $P = \sqrt[3]{0.9371}$.

$$P = 0.9371^{\frac{1}{3}}$$

$$\lg P = \lg 0.9371^{\frac{1}{3}}$$

$$= \frac{1}{3} \lg 0.9371$$

$$= \frac{1}{3} \times \bar{1}.9717$$

$$= (\bar{1}.9717) \div 3$$

$$= (-1 + 0.9717) \div 3$$

$$= (-3 + 2 + 0.9717) \div 3$$

$$= (-3 + 2.9717) \div 3$$

$$= -1 + 0.9906$$

$$= \bar{1}.9906$$

$$\therefore P = \text{antilog } \bar{1}.9906$$

$$= \underline{\underline{0.9786}}$$

Simplifying using indices.

$$\sqrt[3]{0.9371} = 0.9371^{\frac{1}{3}}$$

$$= \left(10^{\bar{1}.9717}\right)^{\frac{1}{3}}$$

$$= 10^{\bar{1}.9717 \times \frac{1}{3}}$$

$$= 10^{\bar{1}.9906}$$

$$= 9.786 \times 10^{-1}$$

$$= \underline{\underline{0.9786}}$$

(iii)Take $P = \sqrt[3]{0.0549}$.

$$
\begin{aligned}
\lg P &= \lg 0.0549^{\frac{1}{3}} \\
&= \frac{1}{3} \lg 0.0549 \\
&= \frac{1}{3} \times \bar{2}.7396 \\
&= (\bar{2}.7396) \div 3 \\
&= (-2 + 0.7396) \div 3 \\
&= (-3 + 1 + 0.7396) \div 3 \\
&= (-3 + 1.7396) \div 3 \\
&= -1 + 0.5799 \\
&= \bar{1}.5799 \\
\therefore P &= \text{antilog } \bar{1}.5799 \\
&= \underline{\underline{0.3801}}
\end{aligned}
$$

Simplifying using indices.

$$
\begin{aligned}
\sqrt[3]{0.0549} &= 0.0549^{\frac{1}{3}} \\
&= \left(10^{\bar{2}.7396}\right)^{\frac{1}{3}} \\
&= 10^{\bar{2}.7396 \times \frac{1}{3}} \\
&= 10^{\bar{1}.5799} \\
&= 3.801 \times 10^{-1} \\
&= \underline{\underline{0.3801}}
\end{aligned}
$$

Now do the following exercise.

Exercise 3.6

1. Find the value using the table of logarithms.

a. $(5.97)^2$ **b.** $(27.85)^3$ **c.** $(821)^3$

d. $(0.752)^2$ **e.** $(0.9812)^3$ **f.** $(0.0593)^2$

2. Find the value using the table of logarithms.

a. $\sqrt{25.1}$ **b.** $\sqrt{947.5}$ **c.** $\sqrt{0.0714}$

d. $\sqrt[3]{0.00913}$ **e.** $\sqrt[3]{0.7519}$ **f.** $\sqrt{0.999}$

3.7 Simplification of expressions involving powers and roots using the table of logarithms

The following example demonstrates how to simplify an expression involving roots, powers, products and divisions (or some of these) using the table of logarithms.

Example 1

Simplify. Give your answer to the nearest first decimal place.

a. $\dfrac{7.543 \times 0.987^2}{\sqrt{0.875}}$ **b.** $\dfrac{\sqrt{0.4537} \times 75.4}{0.987^2}$

a. Take $P = \dfrac{7.543 \times 0.987^2}{\sqrt{0.875}}$

$$
\begin{aligned}
\text{Then } \lg P &= \lg\left(\frac{7.543 \times 0.987^2}{\sqrt{0.875}}\right) \\
&= \lg 7.543 + \lg 0.987^2 - \lg 0.875^{\frac{1}{2}} \\
&= \lg 7.543 + 2 \lg 0.987 - \frac{1}{2} \lg 0.875 \\
&= 0.8776 + 2 \times \bar{1}.9943 - \frac{1}{2} \times \bar{1}.9420 \\
&= 0.8776 + 2 \times \bar{1}.9943 - \frac{\bar{2} + 1.9420}{2} \\
&= 0.8776 + \bar{1}.9886 - (\bar{1} + 0.9710) \\
&= 0.8776 + \bar{1}.9886 - \bar{1}.9710 \\
&= 0.8662 - \bar{1}.9710 \\
&= 0.8952 \\
\therefore \quad P &= \text{antilog } 0.8952 \\
&= 7.855
\end{aligned}
$$

$$\therefore \frac{7.543 \times 0.987^2}{\sqrt{0.875}} \approx \underline{\underline{7.9}} \quad \text{(to the nearest first decimal place)}$$

This simplification can be done by using indices too as follows.

Simplifying using indices.

$$\frac{7.543 \times 0.987^2}{\sqrt{0.875}} = \frac{7.543 \times 0.987^2}{0.875^{\frac{1}{2}}}$$

$$= \frac{10^{0.8776} \times \left(10^{\bar{1}.9943}\right)^2}{\left(10^{\bar{1}.9420}\right)^{\frac{1}{2}}}$$

$$= \frac{10^{0.8776} \times 10^{\bar{1}.9886}}{10^{\bar{1}.9710}}$$

$$= \frac{10^{0.8662}}{10^{\bar{1}.9710}}$$

$$= 10^{0.8662 - \bar{1}.9710}$$

$$= 10^{0.8952}$$

$$= 7.855 \times 10^0$$

$$= 7.855$$

$$\therefore \frac{7.543 \times 0.987^2}{\sqrt{0.875}} \approx 7.9 \quad \text{(to the nearest first decimal place)}$$

b. Take $P = \dfrac{\sqrt{0.4537} \times 75.4}{0.987^2}$

$$\lg P = \lg\left(\frac{0.4537^{\frac{1}{2}} \times 75.4}{0.987^2}\right)$$

$$= \lg 0.4537^{\frac{1}{2}} + \lg 75.4 - \lg 0.987^2$$

$$= \frac{1}{2} \lg 0.4537 + \lg 75.4 - 2 \lg 0.987$$

$$= \frac{1}{2} \times \bar{1}.6568 + 1.8774 - 2 \times \bar{1}.9943$$

$$= \bar{1}.8284 + 1.8774 - \bar{1}.9886$$

$$= 1.7058 - \bar{1}.9886$$

$$= 1.7172$$

$$P = \text{antilog } 1.7172$$

$$= \underline{\underline{52.15}}$$

$$\frac{\sqrt{0.4537} \times 75.4}{0.987^2} \approx \underline{\underline{52.2}} \quad \text{(to the nearest first decimal place)}$$

Simplifying using indices is given below.

$$\frac{\sqrt{0.4537}\times 75.4}{0.987^2} = \left(\frac{0.4537^{\frac{1}{2}}\times 75.4}{0.987^2}\right)$$

$$= \frac{\left(10^{\bar{1}.6568}\right)^{\frac{1}{2}}\quad 10^{1.8774}}{\left(10^{\bar{1}.9943}\right)^2}$$

$$= \frac{10^{\bar{1}.8284}\times 10^{1.8774}}{10^{\bar{1}.9886}}$$

$$= 10^{1.7058-\bar{1}.9886}$$
$$= 10^{1.7172}$$
$$= 52.15$$

$\approx \underline{\underline{52.2}}$ (to the nearest first decimal place)

Exercise 3.7

Use the table of logarithms to compute the value.

a. $\dfrac{8.765\times\sqrt[3]{27.03}}{24.51}$ **b.** $\dfrac{\sqrt{9.18}\times 8.02^2}{9.83}$ **c.** $\dfrac{\sqrt{0.0945}\times 4.821^2}{48.15}$

d. $\dfrac{3\times 0.752^2}{\sqrt{17.96}}$ **e.** $\dfrac{6.591\times\sqrt[3]{0.0782}}{0.9821^2}$ **f.** $\dfrac{3.251\times\sqrt[3]{0.0234}}{0.8915}$

3.8 Applications of logarithms

The table of logarithms can be used to do computations efficiently in many problems that involve products and divisions of numbers. Such an example is given below.

Example 1

The volume V, of a sphere of radius r is given by, $V = \frac{4}{3}\pi r^3$. By taking $\pi = 3.142$ and given that $r = 0.64$ cm, use the table of logarithms to find the volume of the sphere to the nearest first decimal place.

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times 3.142 \times 0.64^3$$
$$\lg V = \lg\left(\frac{4}{3} \times 3.142 \times 0.64^3\right)$$
$$= \lg 4 + \lg 3.142 + 3 \lg 0.64 - \lg 3$$
$$= 0.6021 + 0.4972 + 3 \times \bar{1}.8062 - 0.4771$$
$$= 0.6021 + 0.4972 + \bar{1}.4186 - 0.4771$$
$$= 0.5179 - 0.4771$$
$$= 0.0408$$
$$\therefore \quad V = \text{antilog } 0.0408$$
$$= 1.098$$
$$\approx 1.1 \quad \text{(to the nearest first decimal place)}$$

$\therefore$ The volume of the sphere is 1.1 cm^3.

Exercise 3.8

1. The mass of one cubic centimeter of iron is 7.86 g. Find the mass, to the nearest kilogram, of a cuboidal shaped iron beam, of length, width and depth respectively 5.4 m, 0.36 m and 0.22 m.

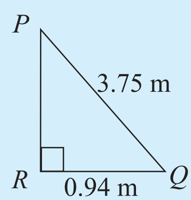
2. Find the value of g, if g is given by $g = \frac{4\pi^2 l}{T^2}$ where, $\pi = 3.142$, $l = 1.75$ and $T = 2.7$

3. A circular shaped portion of radius 0.07 m was removed from a thin circular metal sheet of radius 0.75 m.

(i) Show that the area of the remaining part is $\pi \times 0.82 \times 0.68$.

(ii) Taking π as 3.142, find the area of the remaining part using the table of logarithms.

4. The figure shows a right triangular block of land. If the dimensions of two sides are 3.75 m and 0.94 m, show that the length of PR is $\sqrt{4.69 \times 2.81}$ and find the length of PR in metres to the nearest second decimal place.



3.9 Using a calculator

Logarithms have been used for a long time to do complex numerical computations. However, its use has now been replaced to a great extent by calculators. Computations that can be done using an ordinary calculator is limited. For complex computations one needs to use a scientific calculator. The keyboard of a scientific calculator is much more complex than that of an ordinary calculator.

Evaluating powers using a calculator:

521^3 can be computed by entering $521 \times 521 \times 521$ into an ordinary calculator. However, this can be computed easily using a scientific calculator, by either using the key indicating x^n or by $\boxed{\wedge}$.

Example 1

Find the value of 275^3 using a calculator.
Show the sequence of keys that need to be activated to find 275^3.

$\boxed{2}\ \boxed{7}\ \boxed{5}\ \boxed{x^n}\ \boxed{3}\boxed{=}$ or $\boxed{2}\ \boxed{7}\ \boxed{5}\boxed{\wedge}\boxed{3}\boxed{=}$ 20 796 875

Evaluating roots using a calculator:

You need to use the $\boxed{\text{shift}}$ key, when finding roots. In addition to that, you also need to activate the keys denoted by $\boxed{\sqrt[x]{}}$.

Example 1

Show the sequence of keys that need to be activated to find $\sqrt[4]{2313\ 441}$ using a calculator.

$\boxed{2}\boxed{3}\boxed{1}\ \boxed{3}\ \boxed{4}\ \boxed{4}\ \boxed{1}\boxed{\text{shift}}\boxed{x^n}\ \boxed{4}\ \boxed{=}$

or

$\boxed{2}\boxed{3}\boxed{1}\boxed{3}\boxed{4}\boxed{4}\boxed{1}\ \boxed{x^{1/n}}\ \boxed{4}\boxed{=}$ 39

or

$\boxed{2}\boxed{3}\boxed{1}\boxed{3}\boxed{4}\boxed{4}\boxed{1}\ \sqrt[n]{x}\ \boxed{4}\boxed{=}$

Simplifying expressions involving powers and roots using a calculator:

Show the sequence of keys that need to be activated to find the value of

$$\frac{5.21^3 \times \sqrt[3]{4.3}}{3275}$$

$\boxed{5}\ \boxed{.}\ \boxed{2}\ \boxed{1}\ \boxed{x^n}\ \boxed{3}\ \boxed{\times}\ \boxed{4}\ \boxed{.}\ \boxed{3}\ \boxed{x^{1/n}}\ \boxed{3}\ \boxed{\div}\ \boxed{3}\ \boxed{2}\ \boxed{7}\ \boxed{5}\ \boxed{=}$ 0.070219546

Exercise 3.9

1. Show the sequence of keys that need to be activated to find each of the following values.

a. 952^2 **b.** $\sqrt{475}$

c. 5.85^3 **d.** $\sqrt[3]{275.1}$

e. $375^2 \times \sqrt{52}$ **f.** $\sqrt{4229} \times 352^2$

g. $\dfrac{37^2 \times 853}{\sqrt{50}}$ **h.** $\dfrac{\sqrt{751} \times 85^2}{\sqrt[3]{36}}$

i. $\dfrac{\sqrt{1452} \times 38.75}{98.2}$ **j.** $\dfrac{\sqrt[3]{827.3} \times 5.41^2}{9.74}$

Miscellaneous Exercise

1. Simplify using the table of logarithms. Verify your answer using a calculator.

(i) $\dfrac{1}{275.2}$ (ii) $\dfrac{1}{\sqrt{982.1}}$ (iii) $\dfrac{1}{\sqrt{0.954}}$

(iv) $0.5678^{\frac{1}{3}}$ (v) $0.785^2 - 0.0072^2$ (vi) $9.84^2 + 51.2^2$

2. Find the value of

(i) $\sqrt{\dfrac{a}{b}}$

(ii) $(ab)^2$

when $a = 0.8732$ and $b = 3.168$.

3. In $A = p\left(1 + \dfrac{r}{100}\right)^n$, find the value of A, when $P = 675$, $r = 3.5$ and $n = 3$.

4. A sector with an angle of 73° subtended at the center, was removed from a thin circular sheet.

(i) What fraction of the area of the circle is the area of the sector?

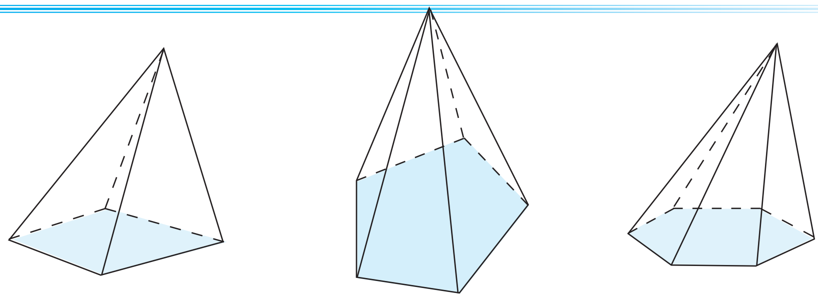
(ii) If the radius of the circle is 17.8 cm, find the area of the sector.

4 Surface area of solids

By studying this lesson you will be able to,

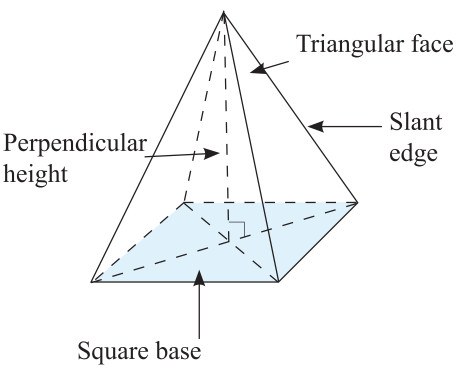
- find the surface area of a right pyramid with a square base,
- find the surface area of a right circular cone, and
- find the surface area of a sphere.

Pyramid



Carefully observe the solid objects in the above figure. Note that their faces are polygons. Of these faces, the horizontal face at the bottom is called the base. All the faces, except the base are of triangular shape. The common point of these triangular faces is called the "apex". A solid object with these properties is called a "pyramid". Note that the bases of the Pyramids shown above are respectively, the shape of a quadrilateral, a pentagon and a hexagon.

Right pyramid with a square base



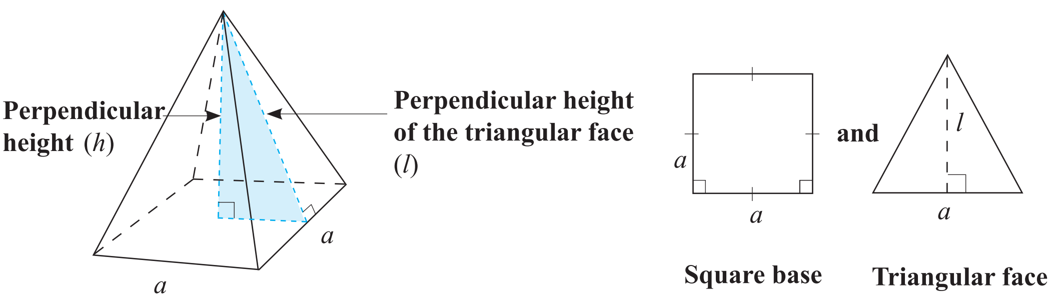
The base of the pyramid in the figure is a square. All the remaining faces are triangular in shape. If the line segment connecting the apex and the midpoint of the square base (that is the intersection point of the two diagonals) is perpendicular to the base, then such a pyramid is called a "**square based right pyramid**". The length of the line segment connecting the apex and the midpoint of the base is called the **perpendicular height** (or simply the height) of the pyramid. The edges of the triangular faces which are not common to the base are called **slant edges**. In this lesson, we will only consider finding the surface area of square based right pyramids.

Note: A tetrahedron can also be considered as a pyramid. All the faces of a tetrahedron are triangular in shape. Any one of the faces can be taken as the base. The concept of "right pyramid" can be defined even when the base is not a square. For example, we can define a right pyramid when the base of a pyramid is a regular polygon, as follows. First note that all the axes of symmetry of a regular polygon pass through a common point, which is called the centroid of the regular polygon. A pyramid, having a base which is a regular polygon, is called a right pyramid, if the line segment connecting the apex and the centroid of the base is perpendicular to the base.
If you study mathematics further, you will learn how to define the centroid, even when the base is not a regular polygon.

An important property of a square based right pyramid is that all its triangular faces are congruent to each other. Therefore, all the triangular faces have the same area. Moreover, note that each triangular face is an isosceles or an equilateral triangle, with one side a side of the square base and the other two sides equal in length.

4.1 Surface area of a square based right pyramid

To find the total surface area of a square based right pyramid we need to add the areas of the base and the four triangular faces.
Suppose the length of a side of the square base is "a" and the perpendicular height of a triangular face is "l".



(There are 4 such faces)

Now, we can find the total surface area as follows.

$$\left.\begin{array}{c}\text{Total surface area of}\\\text{the square based right}\\\text{pyramid}\end{array}\right\} = \left\{\begin{array}{c}\text{Area of the}\\\text{square base}\end{array}\right\} + 4 \times \left\{\begin{array}{c}\text{Area of a triangular}\\\text{face}\end{array}\right\}$$

$$= a \times a + 4 \times \frac{1}{2} \times a \times l$$
$$= a^2 + 2al$$

If the total surface area is A,

$$\boxed{A = a^2 + 2al}$$

Let us now consider some solved examples on the surface area of a square based right pyramid.

Example 1

The base length of a square based right pyramid is 10 cm and the perpendicular height of a triangular face is 15 cm. Find the total surface area of the pyramid.

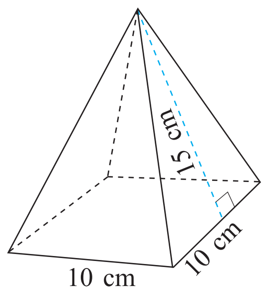
Area of the base $= 10 \times 10$
$= 100$

Area of a triangular face $= \frac{1}{2} \times 10 \times 15$
$= 75$

Area of all four triangular faces $= 75 \times 4$
$= 300$

Total surface area of the pyramid $= 100 + 300$
$= 400$

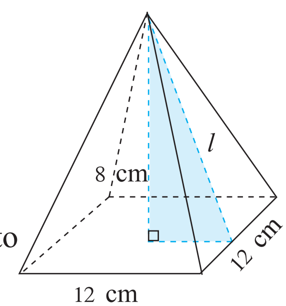
$\therefore$ Total surface area is 400 cm^2.



Example 2

Shown in the figure is a square based right pyramid of perpendicular height 8 cm and base length 12 cm. Find

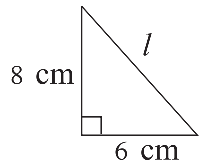
(i) the perpendicular height of a triangular face,
(ii) the area of a triangular face, and
(iii) the total surface area of the pyramid.

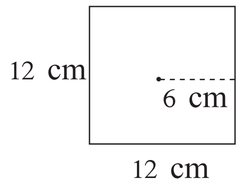


Let us take the perpendicular height of a triangular face to be l cm.
Consider the shaded triangle in the above figure.

By applying Pythagoras' theorem to this triangle,

(i) $l^2 = 8^2 + 6^2$
$= 64 + 36$
$= 100$
$\therefore\ l = \sqrt{100}$
$= 10$





$\therefore$ Perpendicular height of a triangular face is 10 cm.

(ii) Area of a triangular face $= \frac{1}{2} \times 12 \times 10$

$= 60$

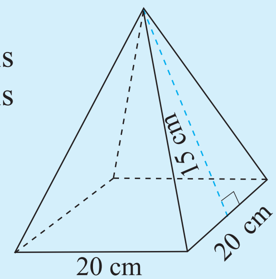
$\therefore$ Area of a triangular face is 60 cm^2.

(iii) Total surface area of the pyramid $= 12 \times 12 + 4 \times 60$
$= 144 + 240$
$= 384$

$\therefore$ Total surface area is 384 cm^2.

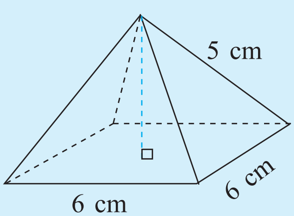
Exercise 4.1

1. The base length of a square based right pyramid is 20 cm and the perpendicular height of a triangular face is 15 cm. Find the total surface area of the pyramid.



2. In a square based right pyramid, the length of a side of the square base is 8 cm and the perpendicular height of a triangular face is 20 cm. What is the surface area of the pyramid?

3.

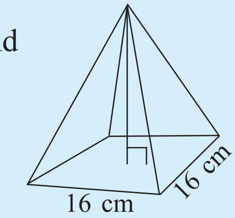


The length of the slant edge of a square based right pyramid is 5 cm, and the length of a side of the base is 6 cm. Find, the total surface area of the pyramid.

4. If the length of a side of the square base of a right pyramid is 20 cm and the perpendicular height is 12 cm, find the total surface area of the pyramid.

5. The base length of a square based right pyramid is 16 cm and the perpendicular height is 6 cm. Find the

(i) perpendicular height of a triangular face.
(ii) the total surface area of the pyramid.



6. Find the total surface area of a square based right pyramid of the length of slant edge is 13 cm, and the side length of the base equal to 10 cm.

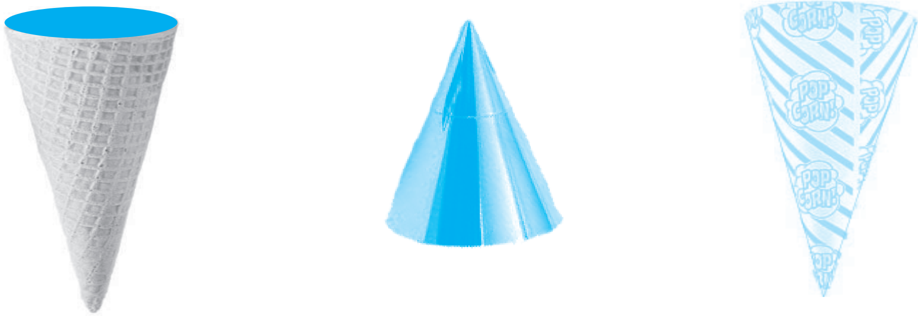
7. The surface area of a square based right pyramid is 2400 cm^2. If the length of a side of the base is 30cm, find
(i) the perpendicular distance from the apex to a side of the base, and
(ii) the height of the pyramid.

8. The area of the fabric that is used to make a tent in the shape of a square based right pyramid, is 80 m^2. Find the height of the tent, if the fabric is not used for the base of the tent and the length of a side of the base is 8cm.

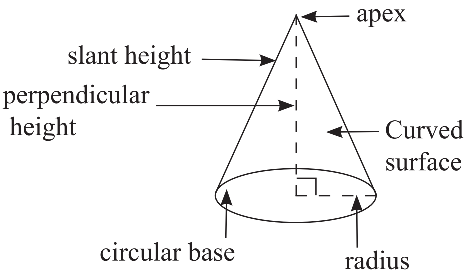
9. The height of a tent in the shape of a square based right pyramid is 4m and the perpendicular height of a triangular face is 5 m. If both the roof and the base of the tent is to be made from fabric, how much material is required?

10. It is required to construct a tent in the shape of a square based pyramid of base length 16 m and height 6 m. Find the fabric needed to construct the tent, also covering the base.

Cone



Shown above are some conical (cone shaped) objects. A cone has a **circular** plane surface and a curved surface. The circular plane surface is called the base of the cone. The point through which all the straight lines drawn on the curved surface pass through is called the "apex" of the cone.



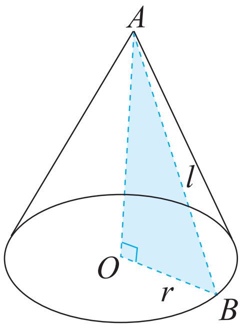
A cone is called a **right circular** cone if the line segment connecting the apex and the centre of the circular base is perpendicular to the circular base of a right circle cone. The radius of the base circle is called the radius of the cone. The length of the line segment connecting the centre of the base and the apex is the perpendicular height of a right circular cone. Moreover, any line segment connecting the apex and a point on the perimeter of the base circle is called a generator of the cone. The length of a generator is called the "slant height " of the cone.

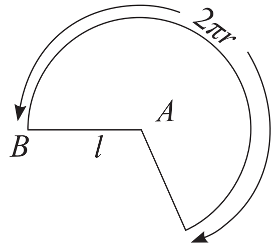
It is customary to use "r" for the radius, "h" for the height and "l" for the slant height of a cone.

4.2 Surface area of a right circular cone

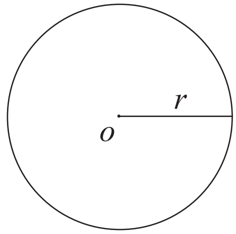
To explain a method to find the surface area of a right circular cone, consider a hollow right circular cone made from a thin sheet. Observe that the base of such a cone is a circular plane surface. Cutting open the curved surface along a generator, gives a lamina, the shape of a circular sector.

Given the radius and the slant height of a right circular cone, one can find surface area of the cone, by finding the area of the circular base and the area of the curved surface. We can use the formula πr^2 to find the area of the circular base. We can find the area of the circular sector as follows.





Curved surface



Circular base

The surface area of the curved surface is equal to the area of the sector that is obtained by cutting it open. Because the arc length of the sector is the circumference of the base circle, the arc length of the sector is equal to $2\pi r$. Also note that the radius of this sector is the slant height "l". Now, as you have learned in the lesson on the perimeter of a circular sector in Grade 10, if the angle of the sector is θ then $\frac{\theta}{360} \times 2\pi l = 2\pi r$. $\therefore \ \theta = \frac{2\pi r \times 360}{2\pi l}$ i . e, $\theta = \frac{360r}{l}$

The area of the sector with the above angle θ (as learnt in grade 10) is $\frac{\theta}{360} \times \pi l^2$.

By substituting θ from the above equation, we get $\frac{360r}{l} \times \frac{\pi l^2}{360}$. Accordingly the area of the curved surface of the cone is πrl.

Now we can add both areas, to get the total surface area of the cone.

$$\text{Total surface area of the cone} = \left\{\begin{array}{c}\text{area of the curved}\\ \text{surface of the cone}\end{array}\right\} + \left\{\begin{array}{c}\text{area of the circular}\\ \text{base}\end{array}\right\}$$
$$= \pi rl + \pi r^2$$

If the total surface area is A

$$\boxed{A = \pi rl + \pi r^2}$$

Let us now consider some solved examples on the surface area of a cone. In this lesson let us take the value of π as $\frac{22}{7}$.

Example 1

Shown in the figure is a solid right circular cone. Its radius is 7 cm and slant height is 12 cm. Find the total surface area of the cone.

Area of the curved surface $= \pi r l$

$= \frac{22}{7} \times 7 \times 12$

$= 264$

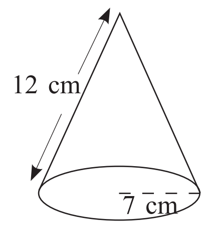
Area of the circular base $= \pi r^2$

$= \frac{22}{7} \times 7 \times 7$

$= 154$

$\therefore$ Total surface area of the cone $= 264 + 154$

$= 418$



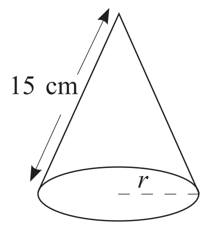
Total surface area of the cone is 418 cm^2.

Example 2

The circumference of the base of a right circular cone is 88 cm and its slant height is 15 cm. Find the area of the curved surface.

Circumference of the circular base = 88 cm

Let us take the radius as r cm.



Then, $2\pi r = 88$

$2 \times \frac{22}{7} \times r = 88$

$r = \frac{88 \times 7}{2 \times 22}$

$r = 14$

Surface area of the curved surface $= \pi r l$

$= \frac{22}{7} \times 14 \times 15$

$= 660$

$\therefore$ Surface area of the curved surface of the cone is 660 cm^2.

Example 3

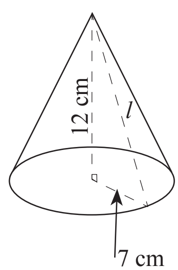
Find,

(i) the slant height,

(ii) the area of the curved surface, and

(iii) the total surface area of the cone,

accurate up to one decimal place, of a cone of radius 7 cm and perpendicular height 12 cm.



Take the slant height of the cone to be l cm.

According to Pythagoras' theorem,

$$\begin{aligned}
\text{(i)}\quad l^2 &= 7^2 + 12^2 \\
&= 49 + 144 \\
&= 193 \\
l &= \sqrt{193} \\
&= 13.8 \quad \text{(Use the division method to find the square root)}
\end{aligned}$$

$\therefore$ The slant height of the cone is 13.8 cm.

(ii) The area of the curved surface $= \pi r l$

$$= \frac{22}{7} \times 7 \times 13.8$$

$$= 303.6$$

$\therefore$ The area of the curved surface is 303.6 cm^2.

(iii) Area of the circular base $= \pi r^2$

$$= \frac{22}{7} \times 7 \times 7$$

$$= 154$$

Total surface area of the cone $= 303.6 + 154$

$$= 457.6$$

$\therefore$ Total surface area of the cone is 457.6 cm^2.

Exercise 4.2

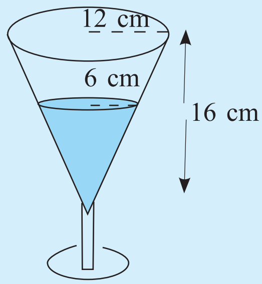
1. Find the area of the curved surface of a right circular cone of base radius 14 cm and slant height 20 cm.

2. Of a right circular solid cone, the radius of the base is 7 cm and the height is 24 cm. Find,
(i) the slant height, and
(ii) the area of the curved surface.

3. If the slant height, of a conical shaped sand pile with a base circumference 44 cm, is 20 cm, find
(i) the radius of the base, and
(ii) the area of the curved surface.

4. Find the total surface area of a right circular cone of base radius 10.5 cm and slant height 15 cm.

5. The slant height of a conical shaped solid object is 14 cm. If the area of the curved surface is 396 cm^2 , find
(i) the radius of the cone, and
(ii) the perpendicular height of the cone.

6. 

Shown in the picture, is a thin glass container in the shape of a cone filled with a juice to half its height. The radius of the glass is 12 cm and its height is 16 cm. Find the area of the region on the glass surface that is in contact with the juice.

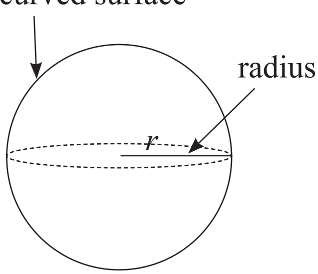
Sphere

shot put

tennis ball

Foot ball

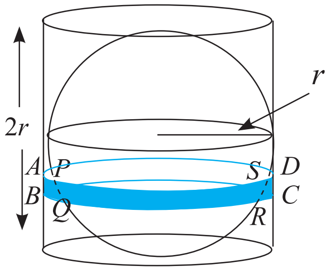
There is no doubt that you know what properties a sphere has. In mathematics, a sphere is defined as the set of points in three dimensional space that lies at a constant distance from a fixed point. The fixed point is called the centre of the sphere and the constant distance from the centre to a point on the sphere is called the radius of the sphere. A sphere has only one curved surface, and has no edges or vertices.



We will usually use "r" to indicate the radius of the sphere.

4.3 Surface area of a sphere

A cylinder with the same radius as the sphere and height equal to the diameter of the sphere is called the **circumscribing cylinder** of the sphere. The sphere will tightly fit in the circumscribing cylinder of the sphere.



The following fact regarding the surface area of a sphere and its circumscribing cylinder was observed by the Greek mathematician Archimedes, who lived around 225 B.C.

When the sphere is inside the circumscribing cylinder, any two planes parallel to the flat circular surfaces of the cylinder will bound equal surface areas on the curved surfaces of the sphere and cylinder.

For example, in the above figure, the area of the curved surface $PQRS$ on the sphere is the same as the area of the curved surface $ABCD$ on the cylinder.

Now, if we apply this fact to the entire cylinder, we see that the surface area of the sphere is equal to the surface area of the curved surface of the cylinder. We can use the formulae $2\pi rh$ to find the surface area of the curved surface of the circumscribing cylinder.

Area of the curved surface of the circumscribing cylinder $= 2\pi r \times 2r$

$= 4\pi r^2$

Therefore, the surface area of the sphere $= 4\pi r^2$

If the surface area is A,

$$\boxed{A = 4\pi r^2}$$

Example 1

Find the surface area of a sphere of radius 7 cm.

$$\begin{aligned}\text{Surface area of the sphere} &= 4\pi r^2\\ &= 4 \times \frac{22}{7} \times 7 \times 7\\ &= 616\end{aligned}$$

$\therefore$ surface area of the sphere is 616 cm^2.

Example 2

The surface area of a sphere is 1386 cm^2. Find the radius of the sphere.

Let r be the radius.

Then, $4\pi r^2 = 1386$

$$4 \times \frac{22}{7} \times r^2 = 1386$$

$$r^2 = \frac{1386 \times 7}{4 \times 22}$$

$$= \frac{441}{4}$$

$$r = \sqrt{\frac{441}{4}}$$

$$= \frac{21}{2}$$

$$= 10.5$$

$\therefore$ radius of the sphere is 10.5 cm.

Exercise 4.3

1. Find the surface area of a sphere of radius 3.5 cm.

2. Find the surface area of a sphere of radius 14 cm.

3. Find the radius of a sphere of surface area 5544 cm^2.

4. Find the (external) surface area of a hollow hemisphere of radius 7 cm.

5. Find the surface area of a solid sphere of diameter 0.5 cm.

6. Find the radius of a solid hemisphere with a surface area of 1386 cm^2.

Summary

- The surface area A of a square based right pyramid, of base length "a" and height "l" is

  $A = a^2 + 2al$

- The surface area A, of a right circular cone of radius r and slant height l is

  $A = \pi rl + \pi r^2$

- The surface area A of a sphere of radius r is

  $A = 4\pi r^2$.

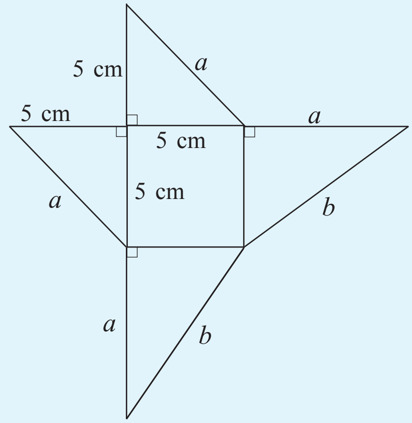
Mixed Exercise

1. Shown below is a net used to make a pyramid.

(i) Find the lengths indicated by a and b.

(ii) Give reasons as to why the resulting pyramid is not a right pyramid.

(iii) Find the total surface area of the pyramid.

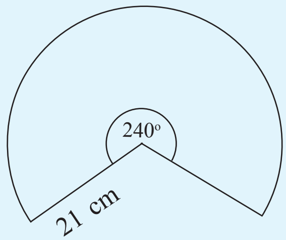


2. A right circular cone was made using a lamina in the shape of the sector shown in the figure.

A circular lamina of the same radius is fixed to the base of the cone.

(i) Find the radius of the cone.

(ii) Find the total surface area of the cone.

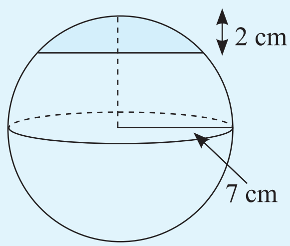


3. The ratio between the slant height and the perpendicular height of a cone is 5 : 4. The radius of the base of the cone is 6 cm.

(i) Find the slant height of the cone.

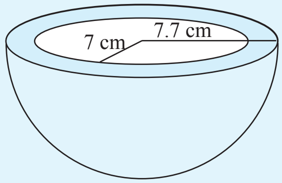
(ii) Find the surface area of the curved surface of the cone.

4.



On a sphere of radius 7 cm, paint was applied from the top downwards, a perpendicular distance of 2 cm. Find the area of the painted region. (Hint: make use of knowledge on the circumscribing cylinder)

5.



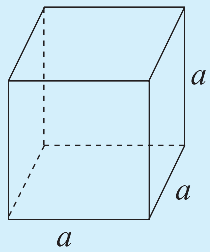
The internal radius of a hemispherical clay pot is 7 cm and the external radius is 7.7 cm. Find the total surface area of the pot.

5 Volume of solids

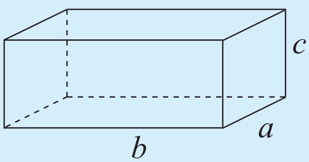
By studying this lesson you will be able to
- compute the volume of a square based right pyramid, right circular cone and a sphere.

Review Exercise

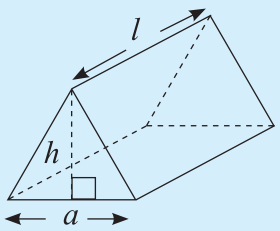
1. Shown below are figures of some solid objects that you have studied before. Complete the given table by recalling how their volume was computed.



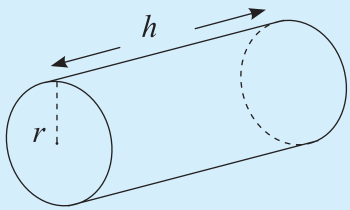
Cube



Cuboid



Triangular prism



Cylinder

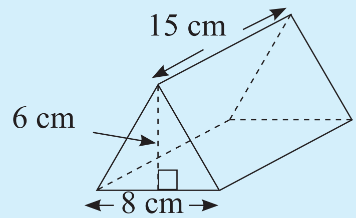
Object	**Cross-sectional area**	**Volume**
Cube		
Cuboid		
Triangular Prism		
Cylinder		

2. Find the volume of a cube of side length 10 cm.

3. Find the volume of a cuboid of length 15 cm, width 10 cm and height 8 cm.

4. Find the volume of a cylinder of radius 7 cm and height 20 cm.

5. Find the volume of the prism shown in the figure.

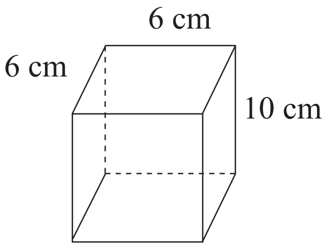


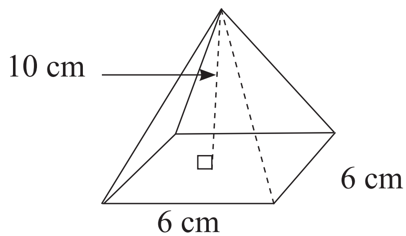
5.1 Volume of a square based right pyramid

Let us do the following activity to construct a formula to find the volume of a square based right pyramid.

Activity

Use a piece of cardboard to construct the hollow cuboid and the hollow pyramid, shown in the following figure. The cuboid has a square base with 6 cm sides and is of height 10 cm. It does not have a top. The right pyramid has a square base, again with 6 cm sides, and the height of the pyramid is also 10 cm. Do not include a base, so that you can fill it with sand.





Completely fill the pyramid with sand and empty it into the cuboid. Find how many times you need to do this to fill the cuboid.

You would have observed in the above activity, that filling the pyramid completely with sand and emptying it into the cuboid thrice, will completely fill the cuboid without any overflow.

Let us consider a square based cuboid with side length a and height h, and a square based right pyramid of side length a and height h.

According to the activity,

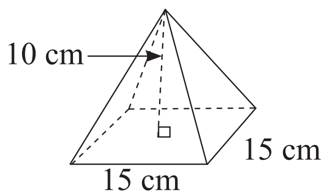
$$\text{Volume of the Pyramid} \times 3 = \text{Volume of the Cuboid}$$

$$\begin{aligned}\therefore\ \text{Volume of the Pyramid} &= \frac{1}{3} \times \text{Volume of the Cuboid}\\ &= \frac{1}{3} \times \text{Area of the Base} \times \text{Perpendicular Height}\\ &= \frac{1}{3} \times (a \times a) \times h\\ &= \frac{1}{3} a^2 h\end{aligned}$$

$$\boxed{\text{Volume of the Pyramid} = \frac{1}{3} a^2 h}$$

Example 1

Find the volume of a square based right pyramid, of height 10 cm and base length 15 cm.



$$\begin{aligned}\text{Volume of the Pyramid} &= \frac{1}{3} a^2 h\\ &= \frac{1}{3} \times 15 \times 15 \times 10\\ &= 750\end{aligned}$$

$\therefore$ Volume of the Pyramid is $750\ \text{cm}^3$.

Example 2

The volume of a pyramid with a square base is $400\ \text{cm}^3$. Find the length of a side of the base, if its height is 12 cm.

Let us take the length of a side of the base to be "a" cm.

$$\text{Volume of the Pyramid} = \frac{1}{3} a^2 h$$

$$\therefore \frac{1}{3} a^2 h = 400$$

$$\therefore \frac{1}{3} a^2 \times 12 = 400$$

$$\therefore 4a^2 = 400$$

$$\therefore a^2 = 100$$

$$= 10^2$$

$$\therefore a = 10$$

$\therefore$ Length of a side of the base is 10 cm.

Exercise 5.1

1. The height of a pyramid with a square base is 9 cm. The length of a side of the base is 5 cm. Find the volume of the pyramid.

2. A square pyramid of height 10 cm has a base of area 36 cm^2. Find the volume of the pyramid.

3. The volume of a square pyramid of height 12 cm is 256 cm^3 . Find the length of a side of the base.

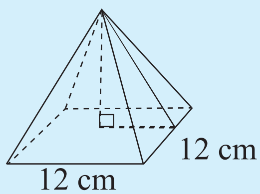
4. The height of a square pyramid is 5 cm. Its volume is 60 cm^3 . Find the area of its base.

5. The volume of a square pyramid is 216 cm^3 . The length of a side of its base is 9 cm. Find the height of the pyramid.

6. The area of the base of a square pyramid is 16 cm^2 and its volume is 80 cm^3. Find the height of the pyramid.

7. The side length of the base of a square pyramid is 12 cm and the slant height is 10 cm. Find

(i) the height, and
(ii) the volume
of the pyramid.



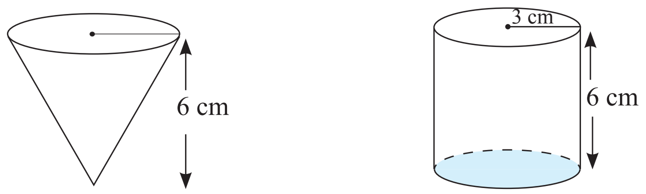
8. The side length of the base of a square pyramid is 10 cm and the slant height is 13 cm. Find
(i) the height, and
(ii) the volume
of the pyramid.

5.2 Volume of a right circular cone

Let us consider constructing a formula for the volume of a right circular cone. Do the following activity using a right circular cone and a right circular cylinder.

Activity

As shown in the figure, using a cardboard, construct a cone without a base, and a cylinder with a base but without a lid, of equal radii and equal height.



Fill the cone completely with sand and empty it into the cylinder. Find how many times you need to do this in order to completely fill the cylinder.

You will be able to observe that, pouring three times from the cone will completely fill the cylinder without any overflow. According to this observation,

$$\text{Volume of the cone} \times 3 = \text{Volume of the cylinder}$$

$$\text{Volume of the cone} = \frac{1}{3} \times \text{Volume of the cylinder}$$

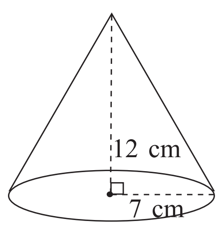
You have learned in a previous lesson that the volume of a cylinder, of radius r and height h, is given by $\pi r^2 h$. Therefore, the volume of a cone of radius r and height h is given by $\frac{1}{3}\pi r^2 h$.

$$\boxed{\text{Volume of the cone} = \frac{1}{3}\pi r^2 h}$$

The value of π is taken as $\frac{22}{7}$ in this lesson.

Example 1

Find the volume of a cone of radius 7 cm and height 12 cm.



$$\begin{aligned}\text{Volume of the cone} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 12 \\ &= 616\end{aligned}$$

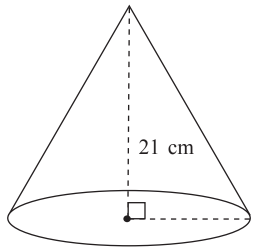
$\therefore$ volume of the cone is 616 cm^3.

Example 2

The circumference of the base of a cone is 44 cm. Its perpendicular height is 21 cm. Find the volume of the cone.

Circumference of the base = 44 cm
Let us take the radius as r centimetres

$$\therefore\ 2\pi r = 44$$
$$2 \times \frac{22}{7} \times r = 44$$
$$\therefore\ r = \frac{44 \times 7}{2 \times 22}$$
$$= 7$$



$\therefore$ radius of the cone is 7 cm.

$$\text{Volume of the cone} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 21$$
$$= 1078$$

$\therefore$ volume of the cone is 1078 cm^3.

Example 3

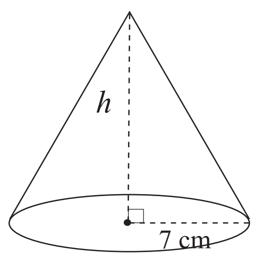
Find,

(i) the height
(ii) the volume

of a cone of radius 7 cm and slant height 25 cm.

Let us indicate the height of the cone by h centimetres. Let us apply Pythagoras' Theorem to the indicated triangle of the cone.

(i)
$$h^2 + 7^2 = 25^2$$
$$h^2 + 49 = 625$$
$$h^2 = 625 - 49$$
$$h = \sqrt{576}$$
$$h = 24$$



$\therefore$ height of the cone is 24 cm.

(ii) volume of the cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 24$$

$$= 1232$$

$\therefore$ volume of the cone is 1232 cm^3.

Example 4

Find the perpendicular height of a cone of radius 3.5 cm and volume 154 cm^3.

Let us indicate the perpendicular height of the cone by h centimetres.

Volume of the cone $= \frac{1}{3}\pi r^2 h$

$$\therefore\ 154 = \frac{1}{3} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times h \qquad \left(\text{because } 3.5 = \frac{7}{2}\right)$$

$$\therefore\ h = \frac{154 \times 3 \times 7 \times 2 \times 2}{22 \times 7 \times 7}$$

$$= 12$$

$\therefore$ perpendicular height of the cone is 12 cm.

Exercise 5.2

1. Find the volume of a cone of radius 7 cm and height 12 cm.

2. Find the volume of a cone of diameter 21 cm and height 25 cm.

3. Find the volume of a cone of slant height 13 cm and base radius 5 cm.

4. Find the volume of a cone of diameter 12 cm and slant height 10 cm.

5. If the height of a cone, of volume 616 cm^3 is 12 cm, find the radius of the cone.

6. The volume of a cone is 6468 cm^3 and its height is 14 cm. Find the diameter of the cone.

7. The circumference of the base of a right cone is 44 cm and its slant height is 25 cm. Compute the following.

(i) Radius of the base

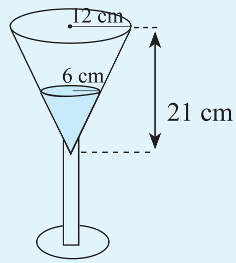
(ii) Height

(iii) Volume

8. The circumference of the base of a conical shaped container is 88 cm and its perpendicular height is 12 cm. Find the volume of the container.

9. How many cones of radius 7 cm and height 15 cm can be made by melting a solid metal cylinder of base radius 14 cm and height 30 cm?

10. A container is of the form of an inverted right circular cone. The radius of the cone is 12 cm and its height is 21 cm. If the container is filled with water to half its height, how much more water is required to fill it completely?

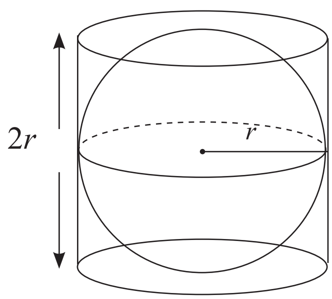


5.3 Volume of a sphere

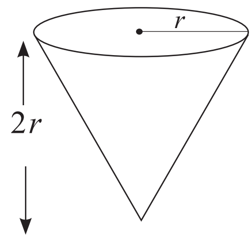
Recall the circumscribing cylinder of a sphere, that we leaned about in the lesson on the surface area of a sphere. Archimedes, the Greek mathematician who explained the surface area of a sphere in terms of the circumscribing cylinder, also explained the volume of a sphere in terms of the volume of the circumscribing cylinder. Let us do the following activity to understand his finding.

Activity

Find a small sphere of radius 3 cm. Using a piece of cardboard, create a cylinder with the same radius as that of the sphere and height equal to the diameter of the sphere. Do not close the two ends of the cylinder. Now carefully insert the sphere into the cylinder.



Take the cylinder, with the sphere inside it. Fill the top part of the cylinder, that is not ocupied by the sphere with sand. Cover top with a piece of cardboad and flip it over and fill the remaining part that is not occupied by the sphere also with sand. Put the total amount of sand that you used to fill the remaining volume of the cylinder after inserting the sphere into the cone that you made earlier. Note that the volume of sand fills the cone completely.



Let us formulate what you have observed in this activity.

Volume of the Circumscribing Cylinder = Volume of the Sphere + Volume of the Cone

Therefore, we can find the volume of the sphere by subtracting the volume of the cone from the volume of the circumscribing cylinder.

Volume of the Sphere = Volume of the Circumscribing Cylinder – Volume of the Cone

$$= \pi r^2 h - \frac{1}{3} \times \pi r^2 h$$

$$= \frac{2}{3} \pi r^2 h$$

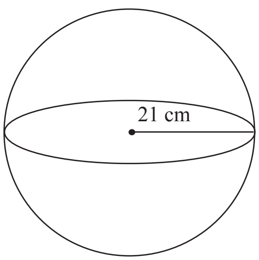
$$= \frac{2}{3} \pi r^2 \times 2r \quad \text{(Because } h = 2r\text{)}$$

$$= \frac{4}{3} \pi r^3$$

$$\boxed{\text{Volume of the Sphere} = \frac{4}{3} \pi r^3}$$

Example 1

Find the volume of a sphere of radius 21 cm.



Volume of the Sphere $= \frac{4}{3} \pi r^3$

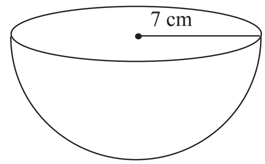
$$= \frac{4}{3} \times \frac{22}{7} \times 21 \times 21 \times 21$$

$$= 38\,808$$

$\therefore$ Volume of the sphere is $38\,808$ cm^3.

Example 2

Find the volume of a solid hemisphere of radius 7 cm.



$$\text{Volume of the hemisphere} = \frac{1}{2} \times \frac{4}{3}\pi r^3$$

$$= \frac{1}{2} \times \frac{4}{3} \times \frac{22}{7} \times 7 \times 7 \times 7$$

$$= 718.67$$

$\therefore$ Volume of the hemisphere is approximately $718.67\ \text{cm}^3$.

Example 3

Find the radius of a small spherical marble, of volume $113\frac{1}{7}\ \text{cm}^3$.
Let us take the radius as r centimetres.

$$\text{Volume of the sphere} = \frac{4}{3}\pi r^3$$

$$\therefore \quad \frac{4}{3}\pi r^3 = 113\frac{1}{7}$$

$$\therefore \quad r^3 = \frac{792}{7} \times \frac{3}{4} \times \frac{7}{22}$$

$$= 27$$

$$= 3^3$$

$$\therefore \quad r = 3$$

$\therefore$ Radius of the sphere is 3 cm.

Exercise 5.3

1. Find the volume of a sphere of radius 7 centimeters.

2. Show that the volume of a sphere of diameter 9 centimeters is $381\frac{6}{7}\ \text{cm}^3$.

3. Find the volume of a spherical celestial body of radius 2.1 kilometers.

4. Find the volume of a hemisphere of radius 10.5 centimeters.

5. The volume of a sphere is $11492\frac{2}{3}$ cubic centimeters. Find the radius of the sphere.

6. Find the radius of the metal ball that is made by using all the metal obtained by melting 8 metal balls, each of radius 7 cm.

7. Show that 32 metal balls of radius 3 cm can be made by melting a solid metal hemisphere of radius 12 cm.

Summary

- The volume V of a square based right pyramid, of base length "a" and height "h" is

$$V = \frac{1}{3}a^2h.$$

- The volume V, of a right circular cone of radius r and height h is

$$V = \frac{1}{3}\pi r^2h.$$

- The volume V of a sphere of radius r is

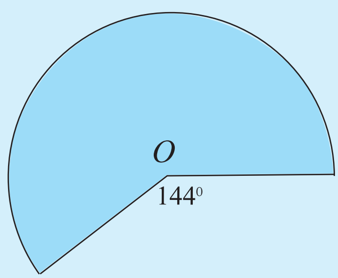
$$V = \frac{4}{3}\pi r^3.$$

Miscellaneous Exercise

1. Solid metal spheres of radius 3 cm were made by melting a metal block. The metal block had a square crossection of 12 cm sides. The length of the metal block was 22 cm. How many metal spheres were made?

2. A solid metal sphere of radius 3.5 cm was melted and casted into a circular cone of the same radius. Find the height of the cone, assuming there was no waste of the metal in the molding process.

3. A cone of slant height r and apex O was constructed using a metal lamina in the shape of the sector shown in the figure. The centre and the radius of the sector are O and r respectively. n pieces of ice in the shape of spheres of radius a are placed in this cone. (the cone is held inverted) If the cone is filled completely with water when all the ice melts show that $125na^3 = 9r^3$.



6

Binomial Expressions

By studying this lesson, you will be able to

expand the cube (third power) of a binomial expression.

You have learned in earlier lessons that, for a binomial expression of the form $x + y$, its square is denoted by $(x + y)^2$, and that what it means is $(x + y)\,(x + y)$, and that when the product is expanded, the expression $x^2 + 2xy + y^2$ is obtained. Moreover, recall that $x^2 - 2xy + y^2$ is obtained when $(x - y)^2$ is expanded.

Do the following exercise to recall what you have learned about the expansion of squares of binomial expressions.

Review Exercise

1. Fill in the blanks.

a. $(a + b)^2 = a^2 + 2ab + ...$
b. $(a - b)^2 = ... - 2ab + b^2$
c. $(x + 2)^2 = x^2 + 4x + ...$
d. $(y + 3)^2 = y^2 + ...\ + 9$
e. $(a - 5)^2 = ...\ - 10a + 25$
f. $(b - 1)^2 = b^2\ .....\ + ...$
g. $(4 + x)^2 = 16 + ...\ .....$
h. $(7 - t)^2 = 49\ .....\ + t^2$
i. $(2x + 1)^2 = 4x^2\ .....\ + 1$
j. $(3b - 2)^2 = ... - 12b\ ......$

2. Expand.

a. $(2m + 3)^2$ **b.** $(3x - 1)^2$ **c.** $(5+2x)^2$
d. $(2a + 3b)^2$ **e.** $(3m - 2n)^2$ **f.** $(2x + 5y)^2$

3. Evaluate the following squares, by writing each as a square of a binomial expression.

a. 32^2 **b.** 103^2 **c.** 18^2 **d.** 99^2

6.1 Cube of a binomial expression

The cube of the binomial expression $a + b$, is $(a + b)^3$. That is, the third power of $(a + b)$. Note that this is the same as multiplying $(a + b)^2$ again by $(a + b)$.

Carefully observe how the following expressions, involving a power of 3, are written.

$$3^3 = 3 \times 3^2 = 3 \times 3 \times 3 = 27$$
$$x^3 = x \times x^2 = x \times x \times x$$
$$(2x)^3 = (2x) \times (2x)^2 = (2x) \times (2x) \times (2x) = 8x^3$$

In a similar way, we can write

$$(x + 1)^3 = (x + 1)(x + 1)^2 = (x + 1)\,(x + 1)\,(x + 1)$$
$$(a - 2)^3 = (a - 2)(a - 2)^2 = (a - 2)\,(a - 2)\,(a - 2)$$
$$(3 + m)^3 = (3 + m)(3 + m)^2 = (3 + m)\,(3 + m)\,(3 + m)$$

The cube of a binomial expression can be expanded in a way similar to how the square of a binomial expression was expanded. It is illustrated in the following example.

Example 1

$$\begin{aligned}(x + y)^3 &= (x + y)\,(x + y)^2 \\ &= (x + y)\,\,(x^2 + 2xy + y^2) \\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 \\ &= \underline{\underline{x^3 + 3x^2y + 3xy^2 + y^3}}\end{aligned}$$

Accordingly, let us remember the following pattern as a formula for the expansion of the cube of the binomial expression $(x + y)$.

$$\boxed{(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3}$$

cube of the first term

three times the product of the square of the first term and the second term.

three times the product of the first term and the square of the second term

cube of the second term

According to this, we can write

$$(m + n)^3 = m^3 + 3m^2n + 3mn^2 + n^3$$

Similarly, we can write $(a+2)^3 = a^3 + 3a^2 \times 2 + 3a \times 2^2 + 2^3$, and this can be further simplified as ,

$a^3 + 6a^2 + 12a + 8$

Now let us consider how the expansion of $(x-y)^3$ is obtained by taking products.

$$\begin{aligned}(x-y)^3 &= (x-y)\,(x-y)^2 \\ &= (x-y)\quad(x^2 - 2xy + y^2) \\ &= x^3 - 2x^2y + xy^2 - x^2y + 2xy^2 - y^3 \\ &= \underline{\underline{x^3 - 3x^2y + 3xy^2 - y^3}}\end{aligned}$$

Now, let us consider how we can obtain the expansion of $(x-y)^3$, using another method.

First, note that we can write $x-y$ as $x + (-y)$. Therefore, we can treat $(x-y)^3$ as an expression of the initial form, by writing it as $\{x+(-y)\}^3$. Let us now consider the expansion of this cube.

$$\begin{aligned}(x-y)^3 = \{x+(-y)\}^3 &= x^3 + 3 \times x^2 \times (-y) + 3 \times x \times (-y)^2 + (-y)^3 \\ &= \underline{\underline{x^3 - 3x^2y + 3xy^2 - y^3}}\end{aligned}$$

Note that we have used the properties $(-y)^2 = y^2$ and $(-y)^3 = -y^3$ in the above simplification.

According to this, we can also write

$$(m-n)^3 = m^3 - 3m^2n + 3mn^2 - n^3$$
$$(p-q)^3 = p^3 - 3p^2q + 3pq^2 - q^3$$

Either method can be used to obtain the expansion of $(x-y)^3$. You may use any method which is easy for you.

Let us now consider how the cube of a binomial expression, involving numbers as well, is expanded.

Example 2

$$\begin{aligned}(x+5)^3 &= x^3 + 3 \times x^2 \times 5 + 3 \times x \times 5^2 + 5^3 \\ &= \underline{\underline{x^3 + 15x^2 + 75x + 125}}\end{aligned}$$

Example 3

$$\begin{aligned}(1+x)^3 &= 1^3 + 3 \times 1^2 \times x + 3 \times 1 \times x^2 + x^3 \\ &= \underline{\underline{1 + 3x + 3x^2 + x^3}}\end{aligned}$$

Example 4

$$\begin{aligned}(y-4)^3 &= y^3 + 3 \times y^2 \times (-4) + 3 \times y \times (-4)^2 + (-4)^3 \\ &= \underline{\underline{y^3 - 12y^2 + 48y - 64}}\end{aligned}$$

or

$$\begin{aligned}(y-4)^3 &= y^3 - 3 \times y^2 \times 4 + 3 \times y \times 4^2 - 4^3 \\ &= \underline{\underline{y^3 - 12y^2 + 48y - 64}}\end{aligned}$$

Example 5

$$\begin{aligned}(5-a)^3 &= 5^3 + 3 \times 5^2 \times (-a) + 3 \times 5 \times (-a)^2 + (-a)^3 \\ &= \underline{\underline{125 - 75a + 15a^2 - a^3}}\end{aligned}$$

or

$$\begin{aligned}(5-a)^3 &= 5^3 - 3 \times 5^2 \times a + 3 \times 5 \times a^2 - a^3 \\ &= \underline{\underline{125 - 75a + 15a^2 - a^3}}\end{aligned}$$

Example 6

$$\begin{aligned}(-2+a)^3 &= (-2)^3 + 3 \times (-2)^2 \times a + 3 \times (-2) \times a^2 + a^3 \\ &= \underline{\underline{-8 + 12a - 6a^2 + a^3}}\end{aligned}$$

Example 7

$$\begin{aligned}(-3-b)^3 &= (-3)^3 + 3 \times (-3)^2 \times (-b) + 3 \times (-3) \times (-b)^2 + (-b)^3 \\ &= \underline{\underline{-27 - 27b - 9b^2 - b^3}}\end{aligned}$$

or

$$\begin{aligned}\left[-1\,(3+b)\right]^3 &= (-1)^3\;(3+b)^3 \\ &= -1\,(3^3 + 3 \times 3^2 \times b + 3 \times 3 \times b^2 + b^3) \\ &= -1\,(27 + 27b + 9b^2 + b^3) \\ &= \underline{\underline{-27 - 27b - 9b^2 - b^3}}\end{aligned}$$

Example 8

Write the expansion of $(x-3)^3$ and verify that $(4-3)^3 = 4^3 - 3^2 \times 4^2 + 3^3 \times 4 - 3^3$

$(x-3)^3 = x^3 - 3^2 \times x^2 + 3^3 \times x - 3^3$

Substituting $x = 4$

Left s. $= (4-3)^3$
$= 1$

Right s. $= 4^3 - 3^2 \times 4^2 + 3^3 \times 4 - 3^3$
$= 1$

Left s. = Right s.

Therefore $(4-3)^3 = 4^3 - 3^2 \times 4^2 + 3^3 \times 4 - 3^3$

Exercise 6.1

1. Fill in the blanks using suitable algebraic terms, symbols (+ or –) or numbers.

a. $(x+3)^3 = x^3 + 3 \times x^2 \times 3 + 3 \times x \times 3^2 + 3^3 = x^3 + \square + \square + 27$

b. $(y+2)^3 = y^3 + 3 \times \square \times \square + 3 \times \square \times \square + 2^3 = y^3 + 6y^2 + \square + \square$

c. $(a-5)^3 = a^3 + 3 \times a^2 \times (-5) + 3 \times a \times (-5)^2 + (-5)^3 = a^3 - \square + \square - 125$

d. $(3+t)^3 = \square + 3 \times \square \times \square + 3 \times \square \times \square + \square = \square + 27t + \square + t^3$

e. $(x-2)^3 = x^3 \square\, 3 \times \square \times \square + 3 \times \square \times \square + (-2)^3 = x^3 \square \square + 12x - \square$

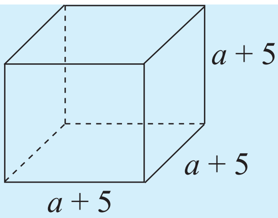
2. Expand.

a. $(m+2)^3$	**b.** $(x+4)^3$	**c.** $(b-2)^3$	**d.** $(t-10)^3$
e. $(5+p)^3$	**f.** $(6+k)^3$	**g.** $(1+b)^3$	**h.** $(4-x)^3$
i. $(2-p)^3$	**j.** $(9-t)^3$	**k.** $(-m+3)^3$	**l.** $(-5-y)^3$
m. $(ab+c)^3$	**n.** $(2x+3y)^3$	**o.** $(3x+4y)^3$	**p.** $(2a-5b)^3$

3. Write as a cube of a binomial expression.

a. $a^3 + 3a^2b + 3ab^2 + b^3$	**b.** $c^3 - 3c^2d + 3cd^2 - d^3$
c. $x^3 + 6x^2 + 12x + 8$	**d.** $y^3 - 18y^2 + 108y - 216$
e. $1 + 3x + 3x^2 + x^3$	**f.** $64 - 48x + 12x^2 - x^3$

4. Shown in the diagram is a cube with the length of each side $(a + 5)$ units. Write an expression for the volume of the cube and expand it.



5. Expand $(x + 3)^3$, and verify the result for the following cases.

(i) $x = 2$
(ii) $x = 4$

.

6. Use the knowledge on cubes of binomial expressions to evaluate the following numerical expressions.

(i) $64 - 3 \times 16 \times 3 + 3 \times 4 \times 9 - 27$
(ii) $216 - 3 \times 36 \times 5 + 3 \times 6 \times 25 - 125$

7. Find the value of each of the following, by writing each as a cube of a binomial expression.

a. 21^3 **b.** 102^3 **c.** 17^3 **d.** 98^3

8. Find the volume of a cube, with each side $2a - 5$ cm, in terms of a.

9. Write $x^3 - 3x^2y + 3xy^2 - y^3$ as a cube and use it to find the value of $25^3 - 3\times 25^2\times 23 + 3 \times 25 \times 23^2 - 23^3$.

7 Algebraic Fractions

By studying this lesson, you will understand

how algebraic fractions are multiplied and divided.

Do the following exercise to revise what you have learned before on adding and subtracting algebraic fractions.

Review Exercise

Simplify.

a. $\frac{a}{5} + \frac{2a}{5}$

b. $\frac{8}{x} - \frac{3}{x}$

c. $\frac{7}{3m} + \frac{3}{4m} - \frac{8}{m}$

d. $\frac{9}{x+2} + \frac{1}{x}$

e. $\frac{1}{m+2} - \frac{2}{m+3}$

f. $\frac{a+3}{a^2-4} + \frac{1}{a+2}$

g. $\frac{2}{x^2-x-2} - \frac{1}{x^2-1}$

h. $\frac{1}{x^2-9x+20} - \frac{1}{x^2-11x+30}$

7.1 Multiplying algebraic fractions

Two algebraic fractions can be multiplied in the same way that two numerical fractions are multiplied. Let us consider the following example.

$\frac{x}{2} \times \frac{x}{3}$

What we mean by performing the multiplication is to express this product as a single fraction.

To perform this multiplication, we multiply the numerators and denominators of the two fractions separately, and obtain a single fraction. That is,

$$\frac{x}{2} \times \frac{x}{3} = \frac{x \times x}{2 \times 3}$$

$$= \frac{x^2}{6}$$

If the terms in the numerator and the denominator can be further simplified, by doing the simplification we can express the answer in the simplest form. These simplifications can be done either before multiplying the fractions or after multiplying the fractions. Let us now consider multiplying two fractions where such simplifications are possible.

Consider $\dfrac{8}{a} \times \dfrac{3}{2b}$.

Here, we can cancel the common factor 2, of the numerator 8 of the first fraction and the denominator $2b$ of the second fraction. We perform this simplification as follows.

$$\frac{8}{a} \times \frac{3}{2b} = \frac{{}^{4}\cancel{8}}{a} \times \frac{3}{{}_{1}\cancel{2}b}$$

Now, by multiplying the expressions in the numerators and denominators of the two fractions separately, we get a single fraction as given below.

$$\frac{8}{a} \times \frac{3}{2b} = \frac{4 \times 3}{a \times b}$$

$$= \underline{\underline{\frac{12}{ab}}} \; .$$

We can also cancel the common factors after multiplying the fractions. Consider the following example.

$$\frac{3}{2a} \times \frac{2b}{3} = \frac{6b}{6a}$$

$$= \frac{b}{a}$$

However, by cancelling the common factors before doing this the multiplication, you can minimise long multiplications and divisions. Therefore, doing this is encouraged.

Observe how the following algebraic expressions are simplified.

Example 1

$$\frac{x}{y} \times \frac{4}{5x}$$

$$= \frac{{}^{1}\cancel{x}}{y} \times \frac{4}{5\cancel{x}_{1}}$$ (Dividing by the common factor x)

$$= \frac{1 \times 4}{y \times 5}$$

$$= \underline{\underline{\frac{4}{5y}}}$$

When multiplying fractions with algebraic expressions in the numerator and the denominator, first factorise the expressions. This is done to cancel the common factors if there are any. Consider the following example.

Example 2

Simplify $\dfrac{2}{x+3} \times \dfrac{x^2+3x}{5}$

$$\frac{2}{x+3} \times \frac{x^2+3x}{5} = \frac{2}{x+3} \times \frac{x(x+3)}{5} \quad \text{(Factorise } x^2+3x \text{)}$$

$$= \frac{2}{x+3} \times \frac{x(x+3)}{5} \quad \text{(Divide by the common factor } x+3\text{)}$$

$$= \underline{\underline{\frac{2x}{5}}}$$

Let us now consider a slightly complex example.

Example 3

Simplify $\dfrac{a^2-9}{5a} \times \dfrac{2a-4}{a^2+a-6}$

$$\frac{a^2-9}{5a} \times \frac{2a-4}{a^2+a-6} = \frac{a^2-3^2}{5a} \times \frac{2(a-2)}{(a+3)(a-2)}$$

$$= \frac{(a-3)(a+3)}{5a} \times \frac{2(a-2)}{(a+3)(a-2)}$$

$$= \underline{\underline{\frac{2(a-3)}{5a}}}$$

because a^2-3^2
$= (a-3)(a+3)$
because a^2+a-6
$= (a+3)(a-2)$

Exercise 7.1

Multiply the following algebraic fractions.

a. $\dfrac{6}{x} \times \dfrac{2}{3x}$

b. $\dfrac{x}{5} \times \dfrac{3}{xy}$

c. $\dfrac{2a}{15} \times \dfrac{5}{9}$

d. $\dfrac{4m}{5n} \times \dfrac{3}{2m}$

e. $\dfrac{x+1}{8} \times \dfrac{2x}{x+1}$

f. $\dfrac{3a-6}{3a} \times \dfrac{1}{a-2}$

g. $\dfrac{x^2}{2y+5} \times \dfrac{4y+10}{3x}$

h. $\dfrac{m^2-4}{m+1} \times \dfrac{m^2+2m+1}{m+2}$

i. $\dfrac{x^2-5x+6}{x^2-1} \times \dfrac{x^2-2x-3}{x^2-9}$

j. $\dfrac{a^2-b^2}{a^2-2ab+b^2} \times \dfrac{2a-2b}{a^2+ab}$

7.2 Dividing an algebraic fraction by another algebraic fraction

Recall how you obtained the answer when dividing one fraction by another fraction. You multiplied the first fraction by the reciprocal of the second fraction. Similarly, when dividing an algebraic fraction by another algebraic fraction, we can instead multiply the first by the reciprocal of the second.

Before we study how algebraic fractions are divided, let us consider the reciprocal of an algebraic fraction.

Reciprocal of an algebraic fraction

Recall the facts we have leaned regarding the reciprocal of a number. If the product of two numbers is 1, then each number is the reciprocal or the multiplicative inverse of the other number.

Because $2 \times \frac{1}{2} = 1$, reciprocal of 2 is $\frac{1}{2}$ and reciprocal of $\frac{1}{2}$ is 2.

Because $\frac{1}{3} \times 3 = 1$, reciprocal of $\frac{1}{3}$ is 3 and reciprocal of 3 is $\frac{1}{3}$.

Because $\frac{4}{5} \times \frac{5}{4} = 1$, reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$ and reciprocal of $\frac{5}{4}$ is $\frac{4}{5}$.

The reciprocal of an algebraic fraction is also described similarly. That is,
if the product of two algebraic fractions is 1, then each algebraic fraction is the reciprocal of the other.

Let us multiply the two algebraic fractions $\frac{5}{x}$ and $\frac{x}{5}$.

$\frac{5}{x} \times \frac{x}{5} = \frac{1}{1} = 1.$

Therefore, $\frac{5}{x}$ is the reciprocal of $\frac{x}{5}$ and $\frac{x}{5}$ is the reciprocal of $\frac{5}{x}$.

Similarly, because

$\frac{x+1}{y} \times \frac{y}{x+1} = 1,$

$\frac{x+1}{y}$ is the reciprocal of $\frac{y}{x+1}$ and $\frac{y}{x+1}$ is the reciprocal of $\frac{x+1}{y}$.

Now it should be clear that, the reciprocal of an algebraic fraction can be obtained

by simply interchanging the numerator and denominator, as you have done with numerical fractions.
Observe the following algebraic fractions and their reciprocals.

algebraic fraction	reciprocal
$\frac{m}{4}$	$\frac{4}{m}$
$\frac{a}{a+2}$	$\frac{a+2}{a}$
$\frac{x-3}{x^2+5x+6}$	$\frac{x^2+5x+6}{x-3}$

Let us now consider how to divid an algebraic fraction by another.

Example 1

Simplify $\frac{3}{x} \div \frac{4y}{x}$

$\frac{3}{x} \div \frac{4y}{x} = \frac{3}{x} \times \frac{x}{4y}$ (Instead of dividing by $\frac{4y}{x}$ we multiply by its reciprocal $\frac{x}{4y}$)

$= \frac{3}{\cancel{x}} \times \frac{\cancel{x}}{4y}$ (Dividing by the common factor x)

$= \frac{3}{4y}$ (Multiplying the numerators and denominators separately)

Let us consider a few more examples.

Example 2

Simplify $\frac{a}{b} \div \frac{ab}{4}$

$\frac{a}{b} \div \frac{ab}{4} = \frac{a}{b} \times \frac{4}{ab}$ (Multiplying by the reciprocal)

$= \frac{\cancel{a}}{b} \times \frac{4}{\cancel{a}b}$ (Cancelling a)

$= \frac{4}{b^2}$

When there are algebraic expressions in both the numerator and the denominator, we can first factor the expressions, so that the common factors can be easily found and cancelled before simplifying.

Look at the following examples.

Example 3

Simplify $\dfrac{3x}{x^2+2x} \div \dfrac{5x}{x^2-4}$

$$\frac{3x}{x^2+2x} \div \frac{5x}{x^2-4} = \frac{3x}{x^2+2x} \times \frac{x^2-4}{5x} \quad \text{(Multiplying by the reciprocal)}$$

$$= \frac{3x}{x(x+2)} \times \frac{(x-2)(x+2)}{5x} \quad \text{(Factoring the expressions and dividing by the common factors)}$$

$$= \underline{\underline{\frac{3(x-2)}{5x}}}$$

Example 4

Simplify $\dfrac{x^2+3x-10}{x} \div \dfrac{x^2-25}{x^2-5x}$

$$\frac{x^2+3x-10}{x} \div \frac{x^2-25}{x^2-5x} = \frac{x^2+3x-10}{x} \times \frac{x^2-5x}{x^2-25}$$

$$= \frac{(x+5)(x-2)}{x} \times \frac{x(x-5)}{(x-5)(x+5)}$$

$$= \frac{x-2}{1}$$

$$= \underline{\underline{x-2}}$$

Exercise 7.2

Simplify the following algebraic fractions.

a. $\dfrac{5}{x} \times \dfrac{10}{x}$

b. $\dfrac{m}{3n} \div \dfrac{m}{2n^2}$

c. $\dfrac{x+1}{y} \div \dfrac{2(x+1)}{x}$

d. $\dfrac{2a-4}{2a} \div \dfrac{a-2}{3}$

e. $\dfrac{x^2+4x}{3y} \div \dfrac{x^2-16}{12y^2}$

f. $\dfrac{p^2+pq}{p^2-pr} \div \dfrac{p^2-q^2}{p^2-r^2}$

g. $\dfrac{m^2-4}{m+1} \div \dfrac{m+2}{m^2+2m+1}$

h. $\dfrac{x^2y^2+3xy}{4x^2-1} \div \dfrac{xy+3}{2x+1}$

i. $\dfrac{a^2-5a}{a^2-4a-5} \div \dfrac{a^2-a-2}{a^2+2a+1}$

j. $\dfrac{x^2-8x}{x^2-4x-5} \times \dfrac{x^2+2x+1}{x^3-8x^2} \div \dfrac{x^2+2x-3}{x-5}$

8 Areas of Plane Figures between Parallel Lines

By studying this lesson you will be able to

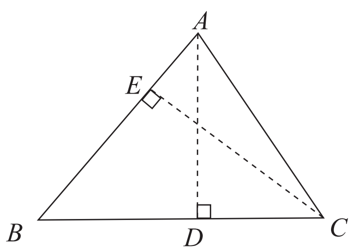
identify the theorems on the relationships between the areas of triangles and parallelograms on the same base and between the same pair of parallel lines, and solve problems related to them.

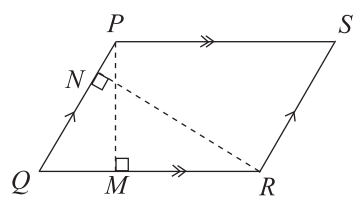
Introduction

You have already learnt about various plane figures and how the areas of certain special plane figures are found. Let us now recall how the areas of triangles and parallelograms are found.

When finding the areas of triangles and parallelograms, the terms altitude and base are used. Let us first recall what these terms mean.

Let us consider the given triangle *ABC* and the parallelogram *PQRS*.





When finding the area of a triangle, any one of its sides can be considered as the base. For example, the side *BC* of the triangle *ABC* can be considered as the base. Then *AD* is the corresponding altitude; that is, the perpendicular dropped from the vertex *A* to the side *BC*.

We know that,

area of triangle $ABC = \frac{1}{2} \times BC \times AD.$

Similarly, if we consider the side *AB* to be the base, the corresponding altitude is *CE*. Accordingly, we can also write,

area of triangle $ABC = \frac{1}{2} \times AB \times CE.$

We can similarly find the area of the triangle *ABC* by taking *AC* as the base and drawing the corresponding altitude from the vertex *B*.

Now let us consider the parallelogram $PQRS$. Here too, the area can be found by considering any one of the sides as the base. If we consider the side QR as the base, the corresponding altitude is the line segment PM. The length of PM is the distance between the two parallel straight line segments QR and PS, the side opposite QR.

We know that,

the area of parallelogram $PQRS = QR \times PM$.

Similarly, if we consider the side PQ as the base, the corresponding altitude is RN. Therefore we can also write,

the area of parallelogram $PQRS = PQ \times RN$.

Note

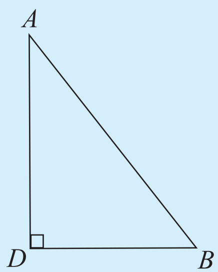
The length of the altitude of a triangle or a parallelogram is also often called the altitude.

To recall what has been learnt earlier regarding finding the areas of parallelograms and triangles, do the following exercise by applying the above facts.

Review Exercise

1. Complete the given table by using the data in each of the figures given below.

(i)



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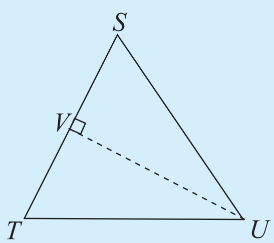
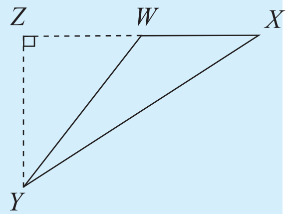
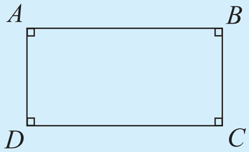


(iii)



(iv)



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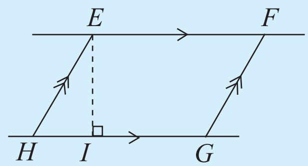


(vi)

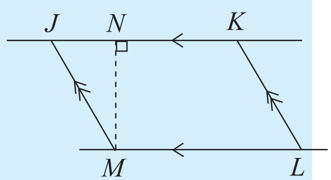


Figure	Base	Corresponding Altitude	Area (As a product of lengths)
(i) Triangle *ABD*			
(ii) Triangle *STU*			
(iii) Triangle *WXY*			
(iv) Rectangle *ABCD*			
(v) Parallelogram *EFGH*			
(v) Parallelogram *JKLM*			

8.1 Parallelograms and triangles on the same base and between the same pair of parallel lines

Let us first see what is meant by parallelograms and triangles on the same base and between the same pair of parallel lines. Consider the following figures.

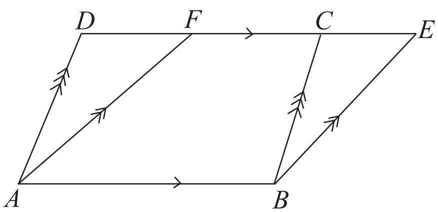


Figure (i)

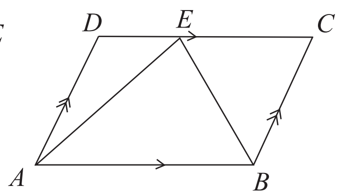


Figure (ii)

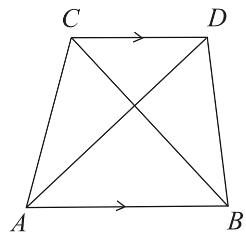


Figure (iii)

Both the parallelograms *ABCD* and *ABEF* in figure (i) lie between the pair of straight lines *AB* and *DE*. What is meant here by the word "between" is that a pair of opposite sides of each of the parallelograms lies on the straight lines *AB* and *DE*. Further, the side *AB* is common to both parallelograms. In such a situation, we say that the two parallelograms are on the same base and between the same pair of parallel lines. Here, the common side *AB* has been considered as the base. It is clear that corresponding to this common base, both the parallelograms have the same altitude. This is equal to the perpendicular distance between the two parallel lines *AB* and *DE*.

Figure (ii) depicts a parallelogram and a triangle which lie on the same base and between the same pair of parallel lines *AB* and *DC*. The parallelogram is *ABCD* and the triangle is *ABE*. The common base is *AB*. Observe that in this case, one side of the triangle lies on one of the parallel lines while the opposite vertex lies on the other line.

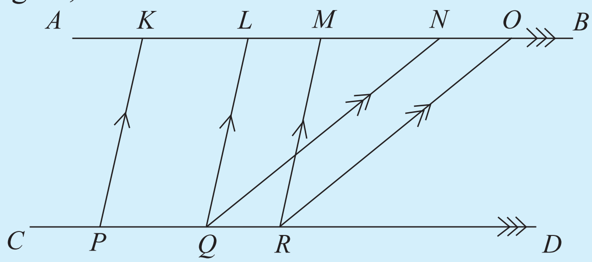
Figure (iii) depicts two triangles on the same base and between the same pair of parallel lines. The two triangles are *ABC* and *ABD*.

Exercise 8.1

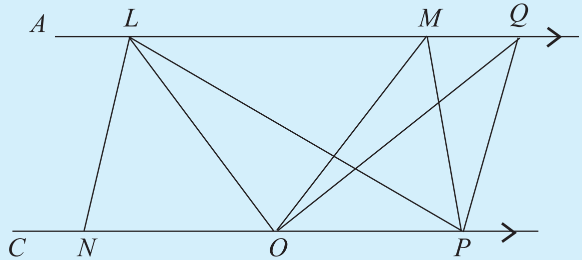
1. Based on the information in the figure,

(i) name four parallelograms.

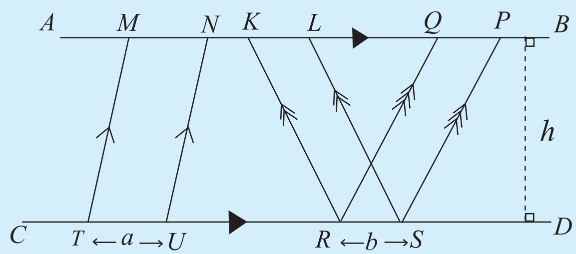
(ii) name the two parallelograms with the same base QR which lie between the pair of parallel lines AB and CD.



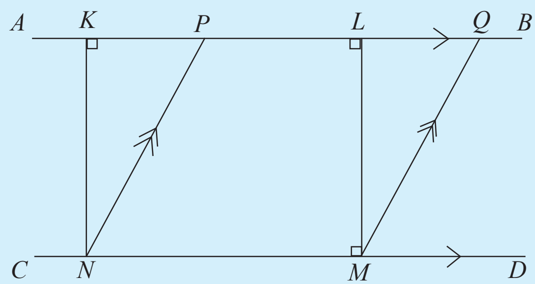
2. Write down all the triangles with the same base OP that lie between the pair of parallel straight lines AQ and CP in the given figure.



3. In the given figure, the perpendicular distance between the pair of parallel straight lines AB and CD is denoted by h and the base lengths of the parallelograms by a and b. Write down the areas of the parallelograms $PQRS$, $KLSR$ and $MNUT$ in terms of these symbols.



4. The rectangle $KLMN$ and the parallelogram $PQMN$ in the given figure lie between the pair of parallel straight lines AB and CD. $NM = 10$ cm and $LM = 8$ cm.



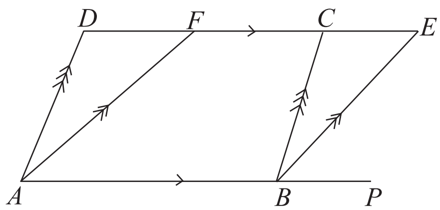
(i) Find the area of the rectangle $KLMN$.

(ii) Find the area of the parallelogram $PQMN$.

(iii) What is the relationship between the area of the rectangle $KLMN$ and the parallelogram $PQMN$?

8.2 The areas of parallelograms on the same base and between the same pair of parallel lines

Next we look at the relationship between the areas of parallelograms on the same base and between the same pair of parallel lines. Consider the given parallelograms.



Let us see whether the areas of the parallelograms $ABCD$ and $ABEF$ are equal.

Observe that,
area of parallelogram $ABCD$ = area of trapezium $ABCF$ + area of triangle AFD
area of parallelogram $ABEF$ = area of trapezium $ABCF$ + area of triangle BEC

Therefore it is clear that, if
the area of triangle AFD = the area of triangle BEC,
then the areas of the two parallelograms will be equal.

In fact, these two triangles are congruent. Therefore their areas are equal. The congruence of the two triangles under the conditions of SAS can be shown as follows.

In the two triangles AFD and BEC,

$AD = BC$ (opposite sides of a parallelogram)
$AF = BE$ (opposite sides of a parallelogram)

Also, since $D\hat{A}B = C\hat{B}P$ (corresponding angles) and $F\hat{A}B = E\hat{B}P$ (corresponding angles), by subtracting these equations we obtain

$D\hat{A}F = C\hat{B}E$.

Accordingly, the two triangles AFD and BEC are congruent under the conditions of SAS.

Therefore we obtain,
area of parallelogram $ABCD$ = area of parallelogram $ABEF$.

We can write this as a theorem as follows.

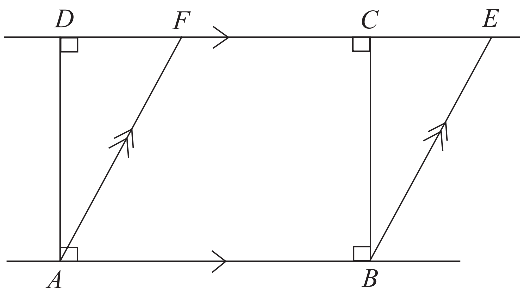
Theorem: Parallelograms on the same base and between the same pair of parallel lines are equal in area.

Now let us obtain an important result using this theorem. You have used the following formula when finding the area of a parallelogram in previous grades and in the above exercise.

Area of a parallelogram = Base $\times$ Perpendicular height

Have you ever thought about how this result was obtained? We can now use the above theorem to prove this result.

The figure depicts a rectangle *ABCD* (that is, a parallelogram) and a parallelogram *ABEF* on the same base and between the same pair of parallel lines. According to the above theorem, their areas are equal.



We know that,

area of the parallelogram *ABEF* = Area of the rectangle *ABCD*

$= AB \times AD$

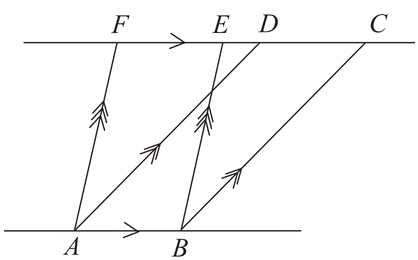
$= AB \times$ perpendicular distance between the two parallel lines

= base of the parallelogram $\times$ perpendicular height

Let us now consider how calculations are done using this theorem.

Example 1

The area of the parallelogram *ABEF* in the figure is 80 cm^2 while $AB = 8$ cm.



(i) Name the parallelograms in the figure that lie on the same base and between the same pair of parallel lines.
(ii) What is the area of the parallelogram $ABCD$?
(iii) Find the perpendicular distance between the parallel lines AB and FC.

Now let us answer these questions.

(i) $ABEF$ and $ABCD$.
(ii) Since the parallelograms $ABEF$ and $ABCD$ lie on the same base AB and between the same pair of parallel lines AB and FC, their areas are equal. Therefore, the area of $ABCD = 80\text{ cm}^2$.
(iii) Let us take the perpendicular distance between the pair of parallel lines as h centimetres.

Then,

area of $ABEF = AB \times h$.

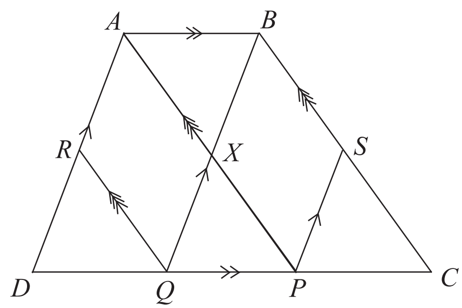
$80 = 8 \times h$

$\therefore\ h = 10$

$\therefore$ the perpendicular distance between the parallel lines is 10 cm.

Now, by considering an example, let us see how riders are proved using this theorem.

Example 2



According to the information in the above figure,

(i) show that $ABQD$ and $ABCP$ are parallelograms.
(ii) show that the parallelograms $ABQD$ and $ABCP$ are of the same area.
(iii) prove that $\Delta\, SPC \equiv \Delta\, RDQ$.
(iv) prove that, area of parallelogram $AXQR$ = area of parallelogram $BXPS$.

(i) In the quadrilateral $ABQD$

$AB // DQ$ (given)

$AD // BQ$ (given)

Since a quadrilateral with pairs of opposite sides parallel, is a parallelogram, $ABQD$ is a parallelogram. Similarly, since $AB // PC$ and $AP // BC$, we obtain that $ABCP$ is a parallelogram.

(ii) Since the parallelograms $ABQD$ and $ABCP$ lie on the same base AB and between the same pair of parallel lines AB and DC, by the above theorem, their areas are equal.

$\therefore$ area of parallelogram $ABQD$ = area of parallelogram $ABCP$.

(iii) In the triangles SPC and RDQ in the figure,

$$S\hat{P}C = R\hat{D}Q \text{ (since } SP \,//\, AD\text{, corresponding angles)}$$

$$S\hat{C}P = R\hat{Q}D \text{ (since } SC \,//\, RQ\text{, corresponding angles)}$$

Further, $AB = PC$ (opposite sides of the parallelogram $ABCP$)

$AB = DQ$ (opposite sides of the parallelogram $ABQD$)

Therefore, $PC = DQ$.

$\therefore \Delta SPC \equiv \Delta RDQ$. (AAS)

(iv) Area of parallelogram $ABQD$ = area of parallelogram $ABCP$ (proved)

Area of ΔRDQ = area of ΔSPC (since $\Delta RDQ \equiv \Delta SPC$)

Therefore,

area of $ABQD$ – area of ΔRDQ = area of $ABCP$ – area of ΔSPC.

Then, according to the figure,

area of trapezium $ABQR$ = area of trapezium $ABSP$.

Therefore,

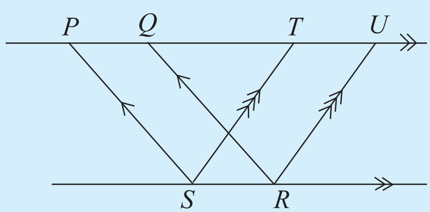
by subtracting the area of the triangle ABX from both sides, we get

area of trapezium $ABQR$ – area of ΔABX = area of trapezium $ABSP$ – area of ΔABX

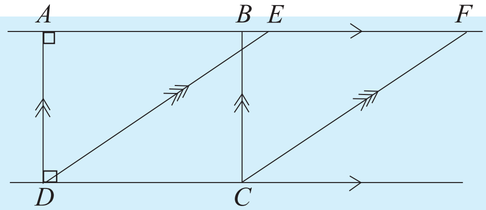
$\therefore$ area of parallelogram $AXQR$ = area of parallelogram $BXPS$.

Exercise 8.2

1. The figure shows two parallelograms that lie between the pair of parallel lines PU and SR. The area of the parallelogram $PQRS$ is 40 cm^2. With reasons, write down the area of the parallelogram $TURS$.



2. A rectangle $ABCD$ and a parallelogram $CDEF$ are given in the figure. If $AD = 7$ cm and $CD = 9$ cm, with reasons, write down the area of the parallelogram $CDEF$.

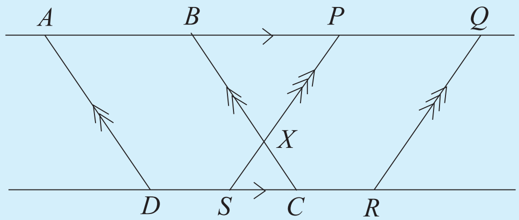


3. The figure shows two parallelograms $ABCD$ and $PQRS$ that lie between the pair of parallel lines AQ and DR. It is given that $DS = CR$.

(i) Show that $DC = SR$.

(ii) Prove that the area of the pentagon $ABXSD$ is equal to the area of the pentagon $PQRCX$.

(iii) Prove that the area of the trapezium $APSD$ is equal to the area of the trapezium $BQRC$.

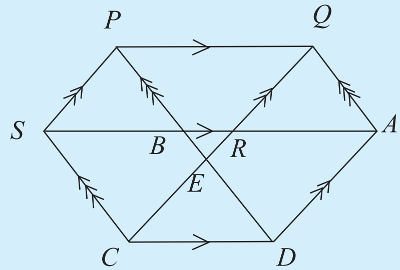


4. Based on the information in the figure,

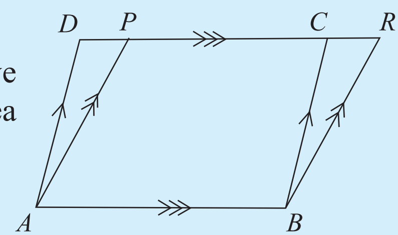
(i) name two parallelograms which are equal in area to the area of the parallelogram $PQRS$.

(ii) name two parallelograms which are equal in area to the area of the parallelogram $ADCR$.

(iii) prove that the area of the parallelogram $PECS$ is equal to the area of the parallelogram $QADE$.



5. Based on the information in the figure, prove that the area of triangle ADP is equal to the area of triangle BRC.



6. Construct the parallelogram $ABCD$ such that $AB = 6$ cm, $D\hat{A}B = 60^0$ and $AD = 5$ cm. Construct the rhombus $ABEF$ equal in area to the area of $ABCD$ and lying on the same side of AB as the parallelogram. State the theorem that you used for your construction.

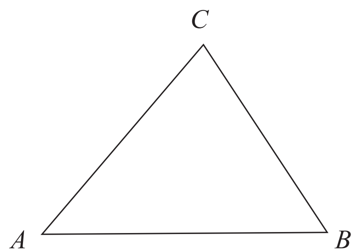
8.3 The areas of parallelograms and triangles on the same base and between the same pair of parallel lines

You have used the following formula in previous grades to find the area of a triangle.

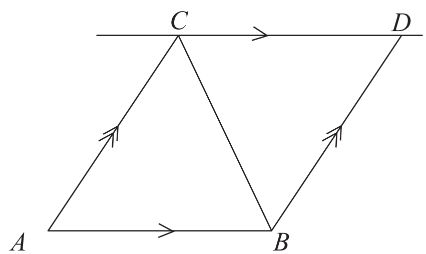
$$\text{Area of the triangle} = \frac{1}{2} \times \text{Base} \times \text{Altitude}$$

Now we will explain why this formula is valid.

Let us consider the following triangle *ABC*.



Now let us draw a line parallel to *AB* through the point *C*, as shown in the figure, and mark the point *D* on this line such that *ABDC* is a parallelogram. In other words let us mark the intersection point of the line drawn through *B* parallel to *AC* and the line drawn through *C* parallel to *AB*, as *D*.



The area of the triangle *ABC* is exactly half the area of the parallelogram *ABDC*. This is because the diagonals of a parallelogram divide the parallelogram into two congruent triangles. We learnt this in the lesson on parallelograms in Grade 10.

Therefore,

$$\begin{aligned}\text{area of triangle } ABC &= \frac{1}{2} \text{ the area of parallelogram } ABDC\\ &= \frac{1}{2} \times AB \times \text{perpendicular distance between } AB \text{ and } CD\\ &= \frac{1}{2} \times AB \times \text{perpendicular height}\end{aligned}$$

We have obtained the familiar formula for the area of a triangle.

Consider again the result that we observed here;

area of triangle $ABC = \frac{1}{2}$ the area of parallelogram $ABDC$.

In section 8.2 of this lesson, we learnt that the areas of parallelograms on the same base and between the same pair of parallel lines are equal. Therefore, in relation to the above figure, the area of any parallelogram that lies on the same base AB and between the same pair of parallel lines AB and CD is equal to the area of $ABDC$. Therefore,

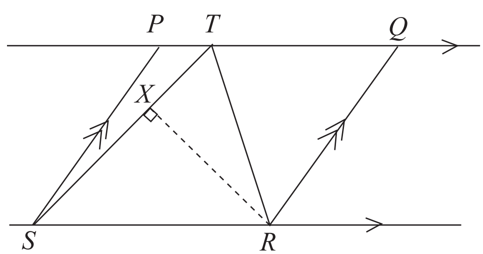
area of triangle $ABC = \frac{1}{2} \times$ (area of any parallelogram with base AB lying between the parallel lines AB and CD).

This result is given below as a theorem.

Theorem: If a triangle and a parallelogram lie on the same base and between the same pair of parallel lines, then the area of the triangle is exactly half the area of the parallelogram.

Let us now consider how calculations are performed using this theorem.

Example 1



The figure illustrates a parallelogram $PQRS$ and a triangle STR on the same base and between the same pair of parallel lines. The area of the parallelogram $PQRS$ is 60 cm^2.

(i) Find the area of the triangle STR. Give reasons for your answer.

(ii) If $ST = 6$ cm, find the length of the perpendicular RX from R to ST.

(i)The parallelogram $PQRS$ and the triangle STR lie on the same base and between the same pair of parallel lines. Therefore the area of triangle STR is half the area of parallelogram $PQRS$.

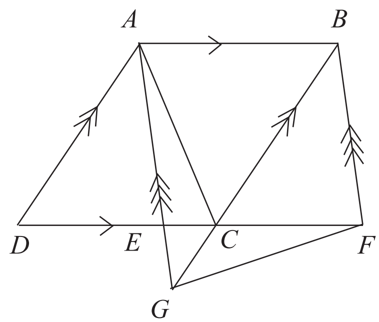
$\therefore$ area of $\Delta\, STR = 30$ cm^2

(ii) Area of $\Delta\, STR = \frac{1}{2} \times ST \times RX$

$$\therefore\; 30 = \frac{1}{2} \times 6 \times RX$$

$$\therefore\; RX = \underline{\underline{10 \text{ cm}}}$$

Example 2



E is a point on the side *DC* of the parallelogram *ABCD*. The straight line drawn through *B* parallel to *AE,* meets *DC* produced at *F.* *AE* produced and *BC* produced meet at *G*.

Prove that,

(i) *ABFE* is a parallelogram.
(ii) the areas of the parallelograms *ABCD* and *ABFE* are equal.
(iii) the area of $\Delta\, ACD$ = the area of $\Delta\, BFG$.

(i) In the quadrilateral *ABFE,*
$AE \,//\, BF$ (data)
$AB \,//\, EF$ (data)
$\therefore$ *ABFE* is a parallelogram (since pairs of opposite sides are parallel)

(ii) The parallelograms *ABCD* and *ABFE* lie on the same base *AB* and between the same pair of parallel lines *AB* and *DF.*
$\therefore$ according to the theorem,
area of parallelogram *ABCD* = area of parallelogram *ABFE*

(iii) The parallelogram *ABCD* and the triangle *ACD* lie on the same base *DC* and between the same pair of parallel lines *AB* and *DC*.
$\therefore$ according to the theorem,
$\frac{1}{2}$ the area of parallelogram *ABCD* = area of triangle *ACD*.

Similarly,

the parallelogram $ABFE$ and the triangle BFG lie on the same base BF and between the same pair of parallel lines BF and AG.
Therefore,

$\frac{1}{2}$ the area of parallelogram $ABFE$ = area of triangle BFG

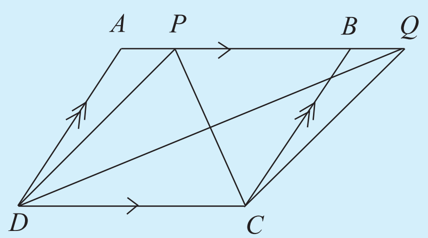
Since, area of parallelogram $ABCD$ = area of parallelogram $ABFE$,

$\frac{1}{2}$ the area of parallelogram $ABCD$ = $\frac{1}{2}$ the area of parallelogram $ABFE$

$\therefore$ area of $\Delta\, ACD$ = area of $\Delta\, BFG$

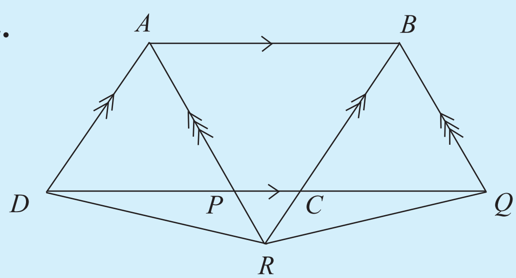
Exercise 8.3

1. The area of the parallelogram $ABCD$ in the figure is 50 cm^2.



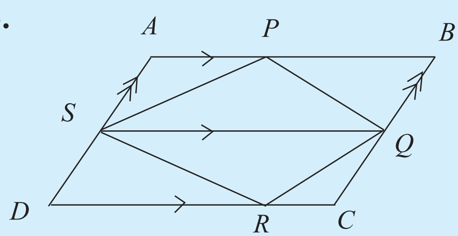
(i) What is the area of triangle PDC?
(ii) What is the area of triangle DCQ?

2.



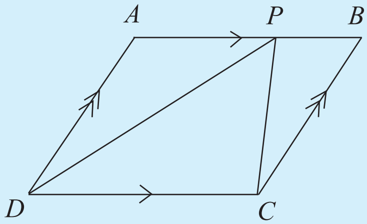
The point P lies on the side DC of the parallelogram $ABCD$. The straight line drawn through B parallel to AP meets DC produced at Q. Further, AP produced and BC produced meet at R. Prove that the area of triangle ADR is equal to the area of triangle BQR.

3.



In the figure, SQ has been drawn parallel to the side AB of the parallelogram $ABCD$, such that it meets the side AD at S and the side BC at Q. Prove that the area of the quadrilateral $PQRS$ is exactly half the area of the parallelogram $ABCD$.

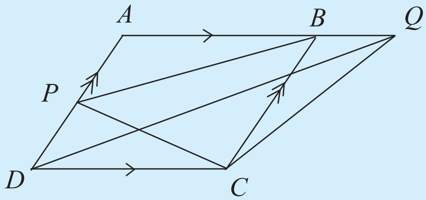
4.



P is any point on the side AB of the parallelogram $ABCD$. Prove that,

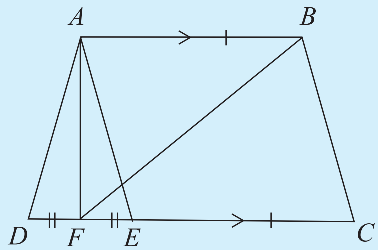
area of ΔAPD + area of ΔBPC = area of ΔDPC

5.



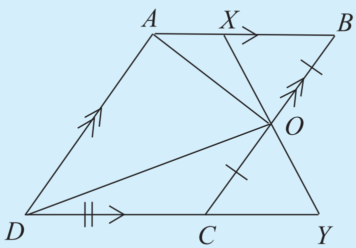
In the figure, the point P lies on the side AD of the parallelogram $ABCD$, and the point Q lies on AB produced. Prove that, area of $\Delta\, CPB$ = area of $\Delta\, CQD$.

6.



$AB//DC$ and $DC > AB$ in the trapezium $ABCD$.
The point E lies on the side CD such that $AB = CE$. The point F lies on the side DE such that the area of the triangle AFE is equal to the area of the triangle ADF. Prove that the area of the trapezium $ABFD$ is exactly half the area of the trapezium $ABCD$.

7.



O is the midpoint of the side BC of the parallelogram $ABCD$ and X is an arbitrary point on AB. Also, XO produced and DC produced meet at Y.

Prove that,

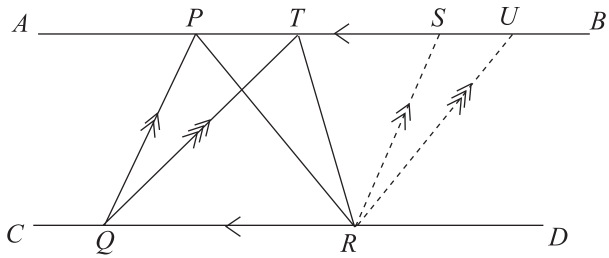
(i) the area of $\Delta\, BOX$ = the area of $\Delta\, COY$

(ii) the area of trapezium $AXYD$ = the area of parallelogram $ABCD$.

(iii) the area of trapezium $AXYD$ is twice the area of triangle ADO.

8.4 Triangles on the same base and between the same pair of parallel lines

Now let us consider the two triangles *PQR* and *TQR* that lie on the same base *QR* and between the same pair of parallel lines *AB* and *CD*.



As discussed in section 8.3 the parallelogram related to the triangle *PQR* is *PQRS*, and the parallelogram related to the triangle *TQR* is *TQRU*.

Since the parallelogram related to the triangle *PQR* is *PQRS,*

area of triangle $PQR = \frac{1}{2}$ the area of parallelogram *PQRS.*

Since the parallelogram related to the triangle *TQR* is *TQRU,*

area of triangle $TQR = \frac{1}{2}$ the area of parallelogram *TQRU.*

However, since the parallelograms *PQRS* and *TQRU* lie on the same base *QR* and between the same pair of parallel lines, by the theorem,
area of parallelogram *PQRS* = area of parallelogram *TQRU.*

$\therefore$ $\frac{1}{2}$ the area of parallelogram $PQRS = \frac{1}{2}$ the area of parallelogram *TQRU*

That is, area of triangle *PQR* = area of triangle *TQR.*

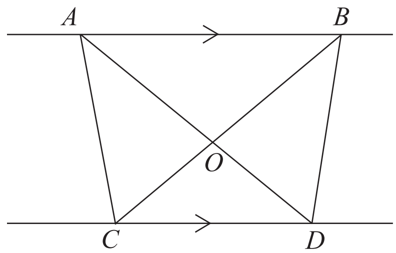
As stated previously, the areas of the two triangles *PQR* and *TQR* which lie on the same base *QR* and between the same pair of parallel lines *AB* and *CD* are equal in area.

Triangles which satisfy the above given conditions in this manner are equal in area. This is stated as a theorem as follows.

Theorem: Triangles on the same base and between the same pair of parallel lines are equal in area.

Let us now consider through the following examples how problems are solved using this theorem.

Example 1



In the given figure, $AB//CD$.

(i) Name a triangle that has the same area as triangle ACD. Write down the theorem that your answer is based on.

(ii) If the area of triangle ABC is 30 cm^2, find the area of triangle ABD.

(iii) Prove that the area of triangle AOC is equal to the area of triangle BOD.

(i) Triangle BCD.

Triangles on the same base and between the same pair of parallel lines are equal in area.

(ii) Area of triangle ABD = 30 cm^2.

(iii) Area of $\Delta\, ACD$ = Area of $\Delta\, BCD$. (On the same base CD and $AB//CD$.)

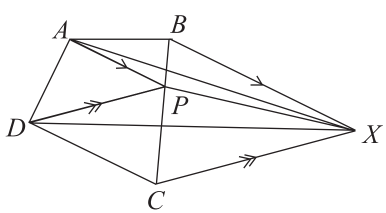
According to the figure, the triangle COD is common to both these triangles. When this portion is removed,

area of $\Delta\, ACD$ – area of $\Delta\, COD$ = area of $\Delta\, BCD$ – area of $\Delta\, COD$

$\therefore$ area of $\Delta\, AOC$ = area of $\Delta\, BOD$

Example 2

The point P lies on the side BC of the quadrilateral $ABCD$. The line drawn though B parallel to AP meets the line drawn through C parallel to DP at X. Prove that the area of triangle ADX is equal to the area of quadrilateral $ABCD$.



Proof: Since the triangles APB and APX lie on the same base AP and between the same pair of parallel lines AP and BX, according to the theorem,

$$\Delta APB = \Delta APX \text{ ——— } ①$$

Similarly, since $DP//CX$,

$$\Delta DPC = \Delta DPX \text{ ——— } ②$$

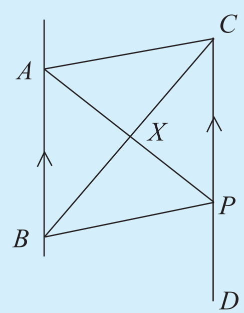
From ① + ②, $\Delta ABP + \Delta DPC = \Delta APX + \Delta DPX.$

Let us add the area of triangle ADP to both sides.

Then, $\Delta ABP + \Delta DPC + \Delta ADP = \Delta APX + \Delta DPX + \Delta ADP$

$\therefore$ area of quadrilateral $ABCD$ = area of the triangle ADX

Exercise 8.4

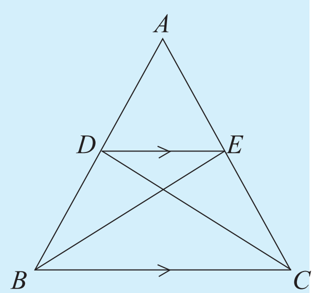


1. The area of triangle ABP which lies between the parallel lines AB and CD in the figure is 25 cm^2.

(i) What is the area of triangle ABC ?

(ii) If the area of triangle ABX is 10 cm^2, what is the area of triangle ACX ?

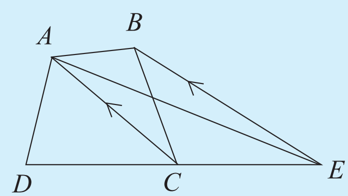
(iii) Explain with reasons what the relationship between the areas of the triangles ACX and BPX is.



2. In the figure, DE is drawn parallel to the side BC of the triangle ABC, such that it touches the side AB at D and the side AC at E.

(i) Name a triangle which is equal in area to the triangle BED.

(ii) Prove that the triangles ABE and ADC are equal in area.

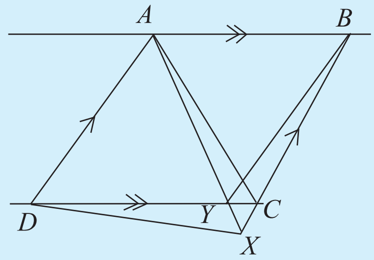


3. The straight line drawn through the point B parallel to the diagonal AC of the quadrilateral $ABCD$, meets the side DC produced at E.

(i) Name a triangle which is equal in area to the triangle ABC. Give reasons for your answer.

(ii) Prove that the area of the quadrilateral $ABCD$ is equal to the area of the triangle ADE.

4. *ABCD* is a parallelogram. A straight line drawn from *A* intersects the side *DC* at *Y* and *BC* produced at *X*.
Prove that,
(i) the triangles *DYX* and *AYC* are equal in area.
(ii) the triangles *BCY* and *DYX* are equal in area.



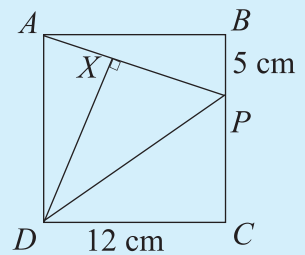
5. The point *Y* lies on the side *BC* of the parallelogram *ABCD*. The side *AB* produced and *DY* produced meet at *X*. Prove that the area of triangle *AYX* is equal to the area of triangle *BCX*.

6. *BC* is a fixed straight line segment of length 8 cm. With the aid of a sketch, describe the locus of the point *A* such that the area of triangle *ABC* is 40 cm^2.

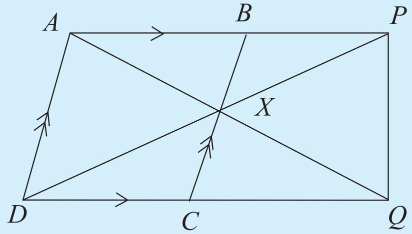
7. Construct the triangle *ABC* such that *AB* = 8 cm, *AC* = 7 cm and *BC* = 4 cm. Construct the triangle *PAB* which is equal in area to the triangle *ABC*, with *P* lying on the same side of *AB* as *C*, and *PA* = *PB*.

Miscellaneous Exercise

1. The length of a side of the square *ABCD* in the figure is 12 cm. The point *P* lies on the side *BC* such that *BP* = 5 cm. Find the length of *DX*.



2. *X* is a point on the side *BC* of the parallelogram *ABCD*. The side *AB* produced and *DX* produced meet at *P* and the side *DC* produced and *AX* produced meet at *Q*. Prove that the area of the triangle *PXQ* is exactly half of the area of the parallelogram *ABCD*.



3. The diagonals of the parallelogram $PQRS$ intersect at O. The point A lies on the side SR. Find the ratio of the area of the triangle POQ to that of the triangle PAQ.

4. $ABCD$ and $ABEF$ are two parallelograms, unequal in area, drawn on either side of AB.
Prove that,
(i) $DCEF$ is a parallelogram.
(ii) the area of the parallelogram $DCEF$ is equal to the sum of the areas of the parallelograms $ABCD$ and $ABEF$.

5. $ABCD$ is a parallelogram. EF has been drawn parallel to BD such that it intersects the side AB at E and the side AD at F.
Prove that,
(i) the triangles BEC and DFC are equal in area.
(ii) the triangles AEC and AFC are equal in area.

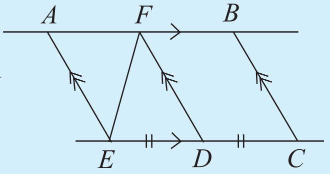
Review Exercise – Term 1

Part 1

1. Simplify $2\sqrt{3} - \sqrt{3}$

2. If $10^{0.5247} = 3.348$ find the value of lg 0.3348.

3. According to the information in the figure, what fraction of the area of *ABCE* is the area of *AFE?*



4. If $A^3 = x^3 - y^3 - 3x^2y + 3xy^2$ express A in terms of x and y.

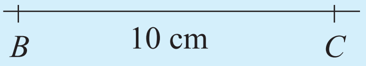
5. A new solid is constructed by pasting together the square bases of two identical square based right pyramids. If the surface area of the new solid is 384 cm^2, find the area of a triangular face of each pyramid.

6. Simplify: $\dfrac{2}{x-1} - \dfrac{1}{1-x}$

7. Evaluate: $\log_3 27 - \log_4 16$

8. The mass of a sphere made of a special type of material is 120 g. If the mass of 1 cm^3 of the material is 4g, find the volume of the sphere.

9. *B* and *C* in the figure are two fixed points that lie 10 cm from each other. Sketch the loci of the point *A* such that the area of the triangle *ABC* is 20 cm^2.

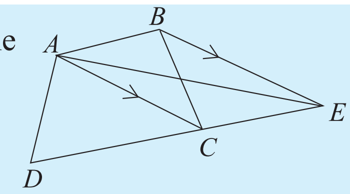


10. If lg 5 = 0.6990 find the value of lg 20.

11. Show that the area of the curved surface of a cylinder of height the length of its diameter, is equal to the surface area of a sphere of the same diameter.

12. Find the value of $\sqrt{20}$ by taking that $\sqrt{5} = 2.23$.

13. Show that the area of the quadrilateral $ABCD$ in the figure is equal to the area of the triangle ADE.



14. Evaluate: $\sqrt{75} \times 2\sqrt{3}$.

15. Simplify: $\dfrac{3x}{x^2 - 1} \times \dfrac{x(x-1)}{3}$

Part II

1. (i) If $x + \dfrac{1}{x} = 3$ then find the value of $x^3 + \dfrac{1}{x^3}$.

(ii) Simplify: $\dfrac{m^2 - 4n^2}{mn(m + 2n)} \div \dfrac{m^2 - 4mn + 4n^2}{m^2n^2}$

2. (i) For what value of x is $2 \lg x = \lg 3 + \lg(2x - 3)$

(ii) If $2 \lg x + \lg 32 - \lg 8 = 2$ determine x.

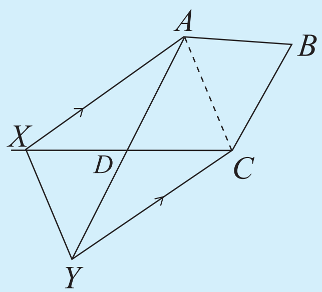
(iii) Find the value without using the logarithms table.

$$\lg_2 \frac{3}{4} - 2 \lg_2 \left(\frac{3}{16}\right) + \lg 12 - 2$$

(iv) Simplify using the logarithms table and give the answer to the nearest second decimal.

$$\frac{\sqrt{0.835} \times 0.75^2}{4.561}$$

3. (a) The side CD of the parallelogram $ABCD$ in the figure has been produced to X. The line drawn through C parallel to AX, meets the side AD produced at Y.



(i) Name a triangle which is equal in area to the triangle AXY. Give reasons for your answer.

(ii) Prove that the area of the triangle XDY is half the area of the parallelogram $ABCD$.

(b) By using only a pair of compasses and a straight edge with a cm/mm scale,

(i) construct the triangle ABC such that $AB = 5.5$ cm, $A\hat{B}C = 60^{\circ}$ and $BC = 4.2$ cm.

(ii) construct the rhombus $ABPQ$ of area twice that of the area of triangle ABC.

4. O is any point on the side BC of the parallelogram $ABCD$. The line drawn through A parallel to DO meets CB produced at P. AO produced meets DC produced at Q.

(i) Based on the above information, sketch a figure and include the given data.

(ii) Write down the relationship between the area of the parallelogram $ABCD$ and the area of the triangle ADO.

(iii) Prove that the area of triangle ABP is equal to the area of triangle BOQ.

5. The base radius and perpendicular height of a solid right circular cone are respectively 7 cm and 12 cm.

(i) Find the volume of the cone.

(ii) If the base radius of the cone is kept fixed and the perpendicular height is doubled, how many times more would the volume of the new cone be than that of the original cone?

(iii) If the perpendicular height is kept fixed and the base radius is doubled, how many times more would the volume of the new cone be than that of the original cone?

ලඝුගණක

மடக்கைகள்

LOGARITHMS

											මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
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12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
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15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
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45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

ලඝුගණක
மடக்கைகள்
LOGARITHMS

											Mean Differences (මධ්‍යන්‍ය අන්තරය / இடை வித்தியாசங்கள்)								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
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56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
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62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Glossary

A

Algebraic Fractions	වීජීය භාග	அட்சரகணிதப் பின்னங்கள்
Area	වර්ගඵලය	பரப்பளவு

B

Bar	වියුති	பிரிகோடு
Base	පාදය	அடி
Binomial Expression	ද්විපද ප්‍රකාශන	ஈருறுப்புக் கோவை

C

Characteristic	පූර්ණාංශය	சிறப்பியல்பு
Circular	වෘත්තාකාර	வட்ட வடிவான
Circumference	පරිධිය	பரிதி
Common denominator	පොදු හරය	பொதுப் பகுதி
Cone	කේතුව	கூம்பு
Cubed	ඝනායිතය	கன
Curved Surface	වක්‍ර පෘෂ්ඨය	வளை மேற்பரப்பளவு

D

Denominator	හරය	Denominator
Division	බෙදීම	வகுத்தல்

E

Entire surds	අඛිල කරණී	முழுமைச் சேடு
Expansion	ප්‍රසාරණය	விரிவு

F

Finite decimals	අන්ත දශම	முடிவுறு தசமம்

I

Indices	දර්ශක	சுட்டி
Infinite decimals	අනන්ත දශම	முடிவில் தசமம்
Integers	නිඛිල	நிறைவெண்கள்
Irrational numbers	අපරිමේය සංඛ්‍යා	விகிதமுறா எண்கள்

L

Least common multiple	කුඩාම පොදු ගුණාකාරය	பொதுமடங்குகளுள் சிறியது
Logarithm	ලඝුගණක	மடக்கை

M

Mantissa	දශමාංශය	தசமக் கூட்டு
Multiplication	ගුණ කිරීම	பெருக்கல்

N

Numerator	ලවය	தொகுதி

P

Parallel lines	සමාන්තර රේඛා	சமாந்தரக் கோடுகள்
Parallelogram	සමාන්තරාස්‍රය	இணைகரம்
Perpendicular height	ලම්බ උස	செங்குத்துயரம்
Power	බලය	வலு
Prism	ප්‍රිස්මය	அரியம்
Pyramid	පිරමීඩය	கூம்பகம்

R

Radius	අරය	ஆரை
Rational numbers	පරිමේය සංඛ්‍යා	
Real numbers	තාත්වික සංඛ්‍යා	மெய் எண்கள்
Reciprocal	පරස්පරය	நிகர்மாறு
Recurring decimals	සමාවර්ත දශම	மீளும் தசமம்
Right circular cone	ඍජු වෘත්ත කේතුව	செவ்வட்டக்கூம்பு
Right pyramid	ඍජු පිරමීඩය	செங்கூம்பகம்

S

Scientific calculator	විද්‍යාත්මක ගණක යන්ත්‍රය	விஞ்ஞானமுறைக் கணிகருவி
Scientific notation	විද්‍යාත්මක අංකනය	விஞ்ஞானமுறைக் குறிப்பீடு
Slant height	ඇල උස	சாய் உயரம்
Sphere	ගෝලය	கோளம்
Square shape	සමචතුරස්‍රාකාර	சதுர வடிவான

Squared	වර්ගායිතය	வர்க்கம்
Surds	කරණි	சேடு
Surface Area	පෘෂ්ඨ වර්ගඵලය	மேற்பரப்பளவு

T

Term	පදය	உறுப்பு
Theorem	ප්‍රමේයය	தேற்றம்
Triangle	ත්‍රිකෝණය	முக்கோணி
Triangular	ත්‍රිකෝණාකාර	முக்கோண வடிவான
Trignometric Ratio	ත්‍රිකෝණමිතික අනුපාත	திரிகோண விகிதங்கள்

V

Volume	පරිමාව	கனவளவு

Sequence of the Lessons

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3. Indices and Logarithms II	06
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5. Volume of Solids	05
6. Binomial Expressions	04
7. Algebraic Fractions	04
8. Areas of Plane Figures between Parallel Lines	12
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09. Percentages	06
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17. Pythagoras Theorem	04
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